PAM and Matched filtering

Previous Summary

$$\begin{tikzcd} s(t) \arrow[r] \arrow[d, "\text{Down Conv}"] & \boxed{\text{B.P. Sys}} \arrow[r] & r(t) \\ s_l(t) \arrow[r] & \boxed{\text{L.P. Eq. System}} \arrow[r] & r_l \arrow[u, "\text{Up Conversion}"'] \end{tikzcd}$$

Down Conversion

Up Conversion

Assumed for analysis

assumed for analysis

Info is not lost
L.P.Eq System fully characterize the B.P.System

B.P → L.P
L.P → B.P

Info is not lost
L.P. Eq System fully characterize the B.P. System

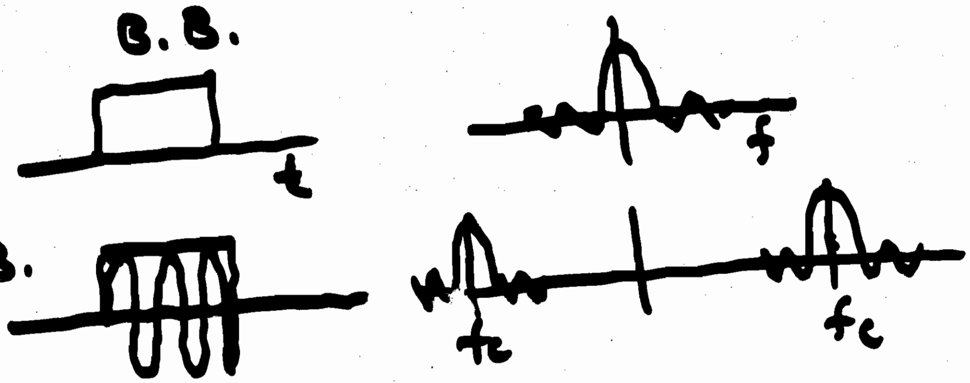
B.B.PAM

B.B.PAM

P.B. PAM

P.B. PAM

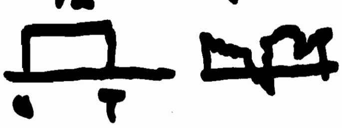
P.A.M.



M-ary

$M = 2^b$ levels e.g.

11	3
10	1
01	-1
00	-3



at rec integrate decide

$\int_0^T \Rightarrow$ filter $\boxed{h(t) = p(t)}$ $p(T-t)$, sample at T

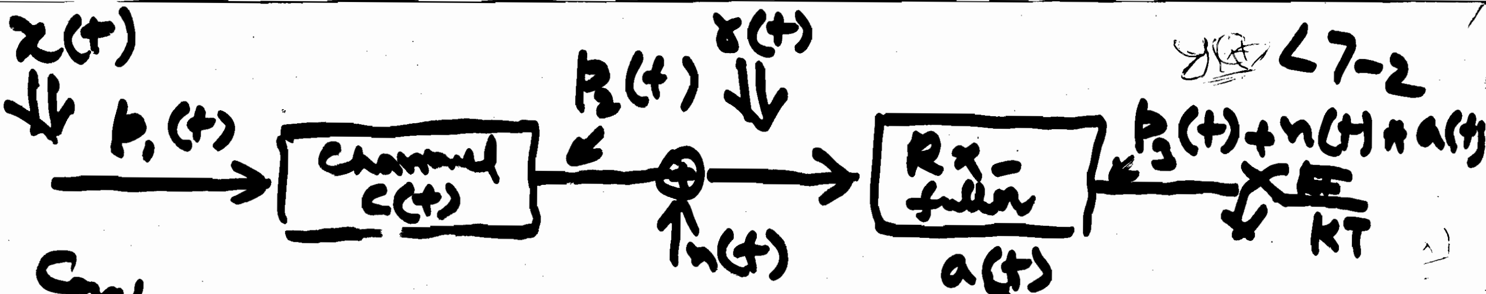
M=2 b level
at receiver integrate
decide
pass through filter
sample at interval T

* pass through filter
* sample at interval T

$$\begin{tikzcd} x(t) \arrow[r] & \boxed{\text{Channel}}_{c(t)} \arrow[r, "r(t)"'] & \boxed{\text{Rx filter}} \arrow[r, "y(t)"] & \text{sample} \arrow[r, "y(kT)"] & {} \end{tikzcd}$$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k\, p(t-kT)$$

say $p(t) = p_1(t)$



Complex pulse

Complex pulse

$$p_2(t) = p_1(t) * c(t)$$

$$p_3(t) = p_2(t) * a(t)$$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k p_1(t-kT)$$

Tx

Rx

$$r(t) = a_k p_1(t-kT) * c(t) + n(t)$$

$$r(t) = a_k p_2(t-kT) + n(t)$$

$$y(t) = r(t) * a(t)$$

$$= \sum_{k=-\infty}^{\infty} a_k p_2(t-kT) * a(t) + n(t) * a(t)$$

[redacted] $n_1(t)$

$$y(t) = \sum_{k=-\infty}^{\infty} a_k p_3(t) + n_1(t)$$

Matched filter

Choose a(t) such that it is matched to get p2(t)
A filter with I.R. a(t)

Matched filter

choose $a(t)$ such that it is matched to get $p_2(t)$

A filter with I.R. $a(t)$

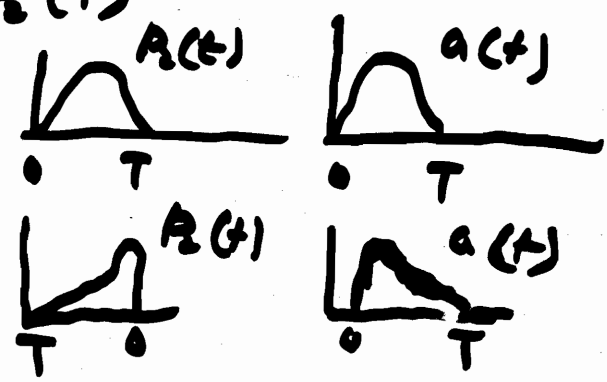
flipped & shifted verion of p2(t)

$$a(t) = p_2^*(T-t)$$

flipped & shifted version of $p_2(t)$

If we demodulate one symbol x(t)

If we demodulate one symbol $x(t)$



$$x(t) = a_k p_1(t-kT)$$

the output

the output

$$r(t) = a_k p_2(t-kT) + n(t)$$

$$y(t) = \int_{-\infty}^{\infty} a(t-\tau)\, r(\tau)\, d\tau$$

$$o/p \quad y(t) = \int_{-\infty}^{\infty} \underbrace{a(t-\tau)}_{\text{o/p of Rx filter}} r(\tau)\, d\tau$$

Convolution of $a(t)$ & $r(t)$

Convolution of a(t) & r(t)

Evaluate at (k+1)th instant (t=(k+1)T)

evaluate at (k+1)th instant

then put $t = T-(k+1)T$

$$\underline{r(\tau)}$$

$$y((k+1)T) = \int_{-\infty}^{\infty} a\left(\underline{(k+1)T} - \tau\right)\left(a_k p_2(\tau - kT) + n(\tau)\right) d\tau$$

$$a(-kT+\tau)\, r(\tau)\, d\tau$$

sampled $\nearrow$ $y((k+1)T)$

sampled version of $y(t)$ at $(k+1)^{th}$ symbol

sampled

sampled version of y(t) at (k+1)th symbol

but we have $\underline{p_2^*(T-t) = a(t)}$ Rx filter

but we have

Rx filter

$$= a_k \int_{-\infty}^{\infty} \underline{p_2^*(\tau - kT)\, p_2(\tau - kT)}\, d\tau + \underbrace{\int_{-\infty}^{\infty} p_2(\tau - kT)\, n(\tau)\, d\tau}_{\text{noise}}$$

noise

$$= a_k \int_{-\infty}^{\infty} \underbrace{|p_2(\tau - kT)|^2}_{E \to \text{Energy of } p(t)} d\tau + n_k$$

fixed

fixed

Energy of p(t)

$$y((k+1)T) = E a_k + n_k$$

y((k+1)T)

Estimate of Tx symbol

$\longrightarrow$ Estimate of Tx Symbol

$$\frac{y((k+1)T)}{E} = a_k + \frac{n_k}{E}$$

a_k = Tx symbol value

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If we know a_k we can know the bit stream
E can be computed

If we know a_k we can know bit stream

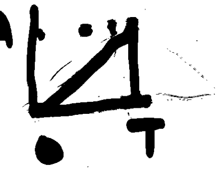
E can be computed

Channel is ideal,
Noise=0

Ex: Channel is Ideal
noise = 0

$$c(t) = \delta(t)$$

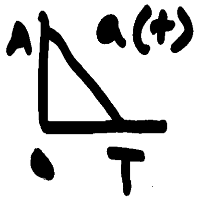
$$p_1(t) =$$



P2(t) = p1(t) * dlta (t)

$$p_2(t) = p_1(t) * \delta(t)$$
$$= p_1(t)$$

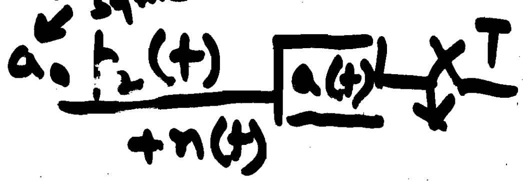
$$a(t) = p_1(T-t)$$



o/p $\quad p_3(t) = p_2(t) * a(t)$

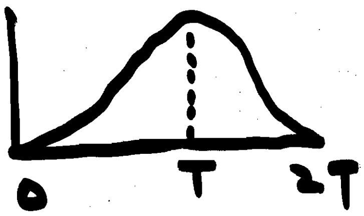
zeroth symb

zeroth symbol



p3(t) is max.at T or multiples of T

$p_3(t)$ is maximum at T & multiples of T

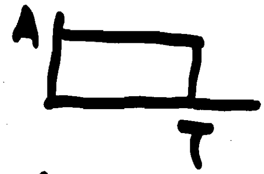


$$y(T) = a_0 p_3(t) + n(t) * a(t) \Big|_{t=T}$$

Sampling at T gives the max SNR.

Sampling at T gives the max. SNR.

Ex: $\quad p_2(t) =$



$$c(t) = \delta(t), \quad a(t) = p_1(T-\tau)$$

[redacted] $p_3(t) = p_1(t) * a(t)$

$$= \int_{-\infty}^{\infty} a(\tau)\, p_1(t-\tau)$$

$$= \int_0^T p(T-\tau)\, p_1(t-\tau)\, d\tau \qquad \begin{matrix} 0 - T \\ \text{other parts} = 0 \end{matrix}$$

max value at $t = T$

o/p filter must be sampled at T
max SNR → matched filter

max value at t=T

O/P filter must be sampled at T
Max SNR------- Matched filter

$$r(t) \rightarrow \boxed{a(t)} \xrightarrow{y(t)} \overset{T}{\times} \rightarrow \hat{a}_k$$

estimate of ak

estimate of a_k

x(tau)

$$r(t) = a_0 p_z(t) + n(t)$$

Assume one symbol

Assume one symbol

convolution of r(t) with a(t) at T

Convolution of r(t) with a(t) at T

$$y(t) = a_0 \int p_z(\tau)\, a(T-\tau)\, d\tau + \int_{-\infty}^{\infty} n(\tau)\, a(T-\tau)\, d\tau$$

$$= y_x(T) + y_n(T)$$

desired O/P

desired o/p

Undesired Noise

undesired noise

SNR Calculation

SNR Calculation

$$SNR = \frac{|y_x(T)|^2}{E\left[|y_n(T)|^2\right]}$$

Energy of the signal

Energy of Signal

Expected Energy of Noise

Expected Energy of Noise

$(a+ib)^2 = (a+ib)(a-ib)$

$$|y_n(T)|^2 = E\left[\left(\int_{-\infty}^{\infty} n^*(t)\, a^*(T-t)\, dt\right)\left(\int_{-\infty}^{\infty} n(\tau)\, a(T-\tau)\, d\tau\right)\right]$$

conjugate

conjugate

a--not random
n-- random,
Expectation for the random only

∵ a → not rand
n → rand
Expect → rand

$$= E\left[\iint E\left[n^*(t)\, n(\tau)\right] a^*(T-t)\, a(T-\tau)\, dt\, d\tau\right]$$

white noise
$\frac{N_0}{2}\delta(t-\tau)$

$$= \frac{N_0}{2} \iint \delta(t-\tau)\, a^*(T-t)\, a(T-\tau)\, dt\, d\tau$$

white Noise
Uncorrelated

$$= \frac{N_0}{2} \int_{-\infty}^{\infty} a^*(T-t)\, a(T-t)\, dt$$

non zero
when tau=t

Expectation of Noise

Expectation of noise

$$= \frac{N_0}{2} \int_{-\infty}^{\infty} |a(T-t)|^2\, dt$$

Signal Energy $\left|\int_{-\infty}^{\infty} p_2(\tau)\, a(T-\tau)\, d\tau\right|^2$

$\underline{a_0 \int_{-\infty}^{\infty} p_2(\tau)\, a(T-\tau)\, d\tau}$

neglecting a_0

$$\text{SNR} = \frac{\left|\int_{-\infty}^{\infty} p_2(\tau)\, a(T-\tau)\, d\tau\right|^2}{\frac{N_0}{2}\int |a(T-t)|^2\, dt}$$

Cauchy Schwarz inequality

$$\left|\int_{-\infty}^{\infty} g_1^*(t)\, g_2(t)\, dt\right|^2 \leq \int_{-\infty}^{\infty} |g_1(t)|^2\, dt \int_{-\infty}^{\infty} |g_2(t)|^2\, dt$$

If we take

$$g_1(t) = a^*(T-t) \qquad |g_1(t)|^2 = |a^*(T-t)|^2$$
$$g_2(t) = p_2(t) \qquad\qquad = |a(T-t)|^2$$

then

$$\left|\int_{-\infty}^{\infty} p_2(\tau)\, a(T-\tau)\, d\tau\right|^2 \leq \int_{-\infty}^{\infty} |p_2(\tau)|^2\, d\tau \int_{-\infty}^{\infty} |a(T-\tau)|^2\, d\tau$$

$$\frac{\left|\int_{-\infty}^{\infty} p_2(\tau)\, a(T-\tau)\, d\tau\right|^2}{\frac{N_0}{2}\int_{-\infty}^{\infty} |a(T-\tau)|^2\, d\tau} \leq \frac{2}{N_0}\int_{-\infty}^{\infty} |p_2(\tau)|^2\, d\tau$$

$$\text{SNR} \leq \frac{2}{N_0}\int_{-\infty}^{\infty} |p_2(\tau)|^2\, d\tau = \frac{2E}{N_0}$$

$$SNR \le \frac{2E}{N_0}$$

constant multiple

It can be equality if g1(t)=c g2(t) (constant multiple)

It can be equality only if $g_1(t) = c\,g_2(t)$

therefore if

Therefore if $g_1(t) = c\,g_2(t)$ when $c = 1$

when c=1

$$g_1(t) = g_2(t)$$

$$\therefore a^*(T-t) = p_2(t)$$

matched filter

matched filter $a(t) = p_2^*(T-t)$

ISI issue

ISI issue

the symbols should not interfere

the symbols should not interfere

If $x(t) = \sum_{k=-\infty}^{\infty} a_k\, p_1(t-kT)$ all symbols

p1

all symbols

$$y(t) = (Tx * Ch + noise) * (Rx\ filter)$$

$$= \big(x(t) * c(t) + n(t)\big) * (a(t))$$

$$= \sum_{m=-\infty}^{\infty} a_m \big(\underline{p_1(t-kT) * c(t) * a(t)}\big) + n(t) * a(t)$$

$$= \sum_{m=-\infty}^{\infty} a_m\, p_3(t-mT) + n(t) * a(t)$$

sampling at kth instant $t = kT$.

sampling at kth instant

t=KT

$$y(kT) = \sum_{m=-\infty}^{\infty} a_m\, p_3(\underbrace{kT-mT}_{(k-m)T}) + \underbrace{n(kT) * a(kT)}_{n_1(kT)}.$$

at ~~[illegible]~~ $m = k$

$$y(kT) = \underbrace{a_k\, p_3(0)}_{\text{kth symbol}} + \underbrace{\sum_{\substack{m=-\infty \\ m \neq k}}^{\infty} a_k\, p_3(kT-mT)}_{\text{ISI}} + n_1(kT)$$

kth symbol

ISI $\Leftarrow$ Effect of other symbols on kth sym.

kth symbol

ISI

Effect of other symbols on kth symbol

let $p_3(t) \rightarrow g(t)$

we need ISI $= 0$

$$\underline{a_k} * g(kT) = a_k \cdot g(0)$$

a_k ↓ seq, $g(kT)$ ↓ seq, a_k ↓ seq., $g(0)$ ↳ constant

at kth sample the convolution should have only first term & all ISI terms should be zero.

at kth sample the convolution
should have only first term
& ISI terms=0

It would only be true if $g(kT)$ is delta sequence i.e. delta at k.

$$g(kT) = \delta(k)$$

⟶ sampled version of $g(t)$

It would be true only if
g(kT) is delta sequence
i.e. delta at k

sampled version
of g(t)

$F[\delta(t)] = 1$

$$g(t) \sum_{k=-\infty}^{\infty} \delta(t-kT) = \delta(t)$$

~~shift~~ shifted at k by T.

$$\text{F.T.} \quad G(f) * \sum_{m=-\infty}^{\infty} \delta\left(f - \frac{m}{T}\right) = 1$$

$\boxed{\delta \text{ at freqs } 1/T \text{ apart}}$

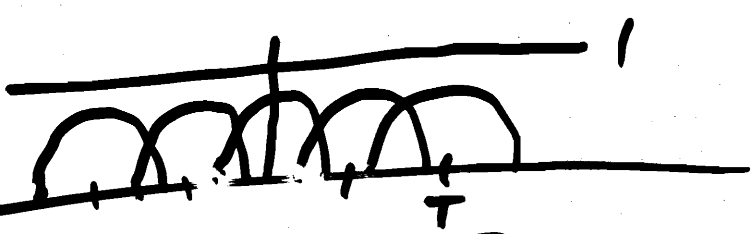
Conv in F.D. → will. :δ

$$\boxed{\sum_{m=-\infty}^{\infty} G\left(f - \frac{m}{T}\right) = 1}$$

condition for ISI = 0 iff

Conv in F.D.

IFF condition for ISI=0



$g(t) = p_3(t)$

shift spectrum at freq multiple of T and add them gives you constant spectrum

shifting spectrum at f multiple of T and adding them
gives you constant spectrum

To achieve zero ISI, we must choose g(t) in a way that its spectrum (shifted, all T) is constant

To achıeve ISI=0, we must choose g(t) ın a way that the spectrum (shifted at all T) is constant

Let $G(f) = 0$ for $f > B$, $f < -B$ — bandlimited

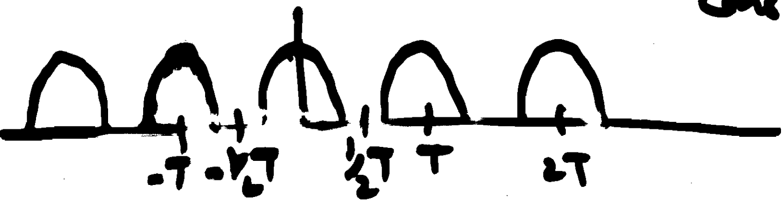
bandlimited

B< 1/2T

(i) If $B < 1/2T$

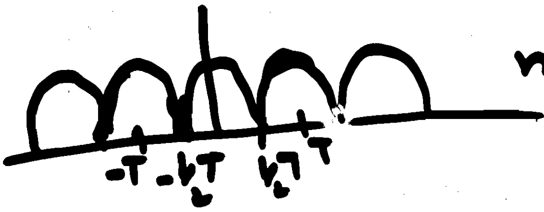
not possible constant spectrum

not possible
constant spectrum

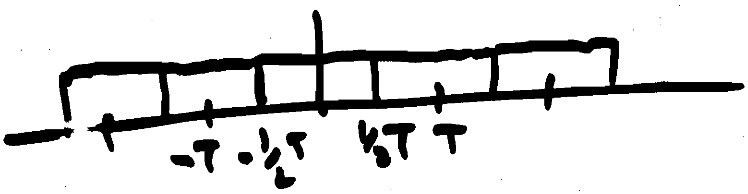


not possıble
constant spectrum

(ii) If $B = 1/2T$



not possible constant spectrum

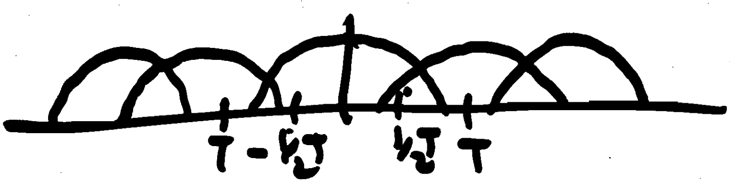


→ ISI = 0 only if P(t) = ⊓

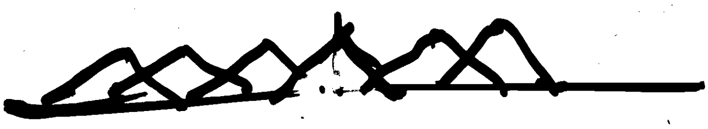
ISI=0 only if P(t) = |_____|

(iii) If $B > 1/2T$

numerous possıbılıty
of choice



numerous possibility of choice



Nyquist criteria is required by pulse to have zero ISI.

Nyquıst crıterıa ıs requıred by the pulse to have ISI = 0