Information Theory

Information theory

Idea about: possible to achieve in point to point Communication Channel.

Idea about : what is possible to achieve in point to point communication channel

How much a signal can be compressed

How much signal can be compressed

Source ——→ Destination
Info

Source --------------Destination
Info

Signal ——→ R. Process

Receiver Processing

Source → [Channel] → Dest

Source ------Channel--------Destination

Processing : to Transmit max info.

Processing to Transmit Maximum info

Source - Processing - Channel - Receiver - Destination

Source → Processing → Channel — Reciever — Destination

Tx

Max. transfer rate
Min. compress rate

Max transfer rate
Min Compress rate

Processing SC
CC

Processing < SC / CC

SC → to compress the source data
CC → transmit through Ch in Efficient way (Max)

Soucre coding --- to compress the source data
Channel coding --- to transmit through channel in an efficient (max) way

LS-2

Source Coding

Example — Horse Race / Car Race

8 Cars -3 bit coding
Avg. code length 3 bit

8 Cars
3bit coding (fix)
Avg :- 3bit

$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{64}, \frac{1}{64}, \frac{1}{64}$

log2(1/prob) = # of bits

$\log_2\left(\frac{1}{prob}\right) = \#$ of bits

weighted avg.

Weighted Average

$\underset{1}{0},\ \underset{2}{10},\ \underset{3}{110},\ \underset{4}{1110},$ 111100, 111101, 111110, 111111 — 6

Most frequent events
should be assigned less no. of bits

Most freq events should be assigned less no of bits

Fewer bits → Freq Event

$\frac{1}{2}\times1 + \frac{1}{4}\times2 + \frac{1}{8}\times3 + \frac{1}{16}\times4 + \left(\frac{1}{64}\times6\right)\times4$

= ~~2 bits~~ 2 bits

ENTROPY
Set of events
r.v. x € {1,2,....M}
with probs
p(1), P(2)....
P(M)
then the info contained
in events is
log(1/p(1)),
log(1/P(2))....
log(1/P(M))

Set of Events r.v. $x \in \{1, 2 \ldots M\}$ with probab $p(1), p(2) \ldots p(M)$

Then info contained in events is $\log\frac{1}{p(1)}, \log\frac{1}{p(2)} \ldots \log\frac{1}{p(M)}$

← Entropy

$$H(x) = \sum_i p(i) \log\frac{1}{p(i)}$$

Ex: $x = \begin{cases} 1 & p \\ 0 & p-1 \end{cases}$

$$H(x) = p\log\frac{1}{p} + (1-p)\log\left(\frac{1}{1-p}\right)$$
$$= -p\log p - (1-p)\log(1-p)$$

If $p=0$
$H(p) = \begin{cases} 0 & p = 0/1 \\ 1 & p = 1/2 \end{cases}$

min # of bits required to represent the r.v. on average is H(x)

min # of bits required to represent the r.v. on average is $H(x)$

Joint Entropy

Joint Entropy $H(x,y) = \sum_x \sum_y p(x,y) \log p(x,y)$

Ex r.v. $x = \begin{cases} a & 1/2 \\ b & 1/4 \\ c & 1/8 \\ d & 1/8 \end{cases}$

a	1/2
b	1/4
c	1/8
d	1/8

If the code length is set equal to the H(x)
i.e. the info. contained in outcomes, the
avg. no. of bits in the code would be minimized

If the code length is set equal to the $H(x)$ i.e. info contained in outcomes, the average no of bits in the code would be minimized

Entropy depends on the prob. dist. of r.v.x

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1. $H(x) \geq 0$

2. $H(x) \leq L_{av} < H(x) + 1$

$L_{av} \rightarrow$ Avg length of code.

Avg. length of code

Shannon Theorem Source Coding Theorem

If $H(x)$ is entropy of r.v. x then we can represent x by $H(x)$ no. of bits on average, if we apply block coding

If source generates i.i.d r.v.s $X_1, X_2 \ldots X_n$ then Total info contained would be sum of info in all $x_1 \ldots\ldots x_n$

$$n H(x) \leq L_M \leq n H(x) + 1$$

for M nos of symbols.

$$H(x) \leq \frac{L_M}{n} \leq H(x) + \frac{1}{n}$$

Joint Entropy

If we have two r.v.s X & Y then the joint entropy would be $H(X,Y) = \sum_x \sum_y p(x,y) \log \frac{1}{p(x,y)}$

$$\neq H(x) + H(y)$$

Conditional Entropy

$$H(Y/X) = \sum_x p(x) H(Y/X = x)$$

$$H(X,Y) = H(X) + H(Y/X)$$

Mutual Information

$$I(X;Y) \triangleq \sum_x \sum_y p(x,y) \log \frac{p(x,y)}{p(x)\, p(y)}$$

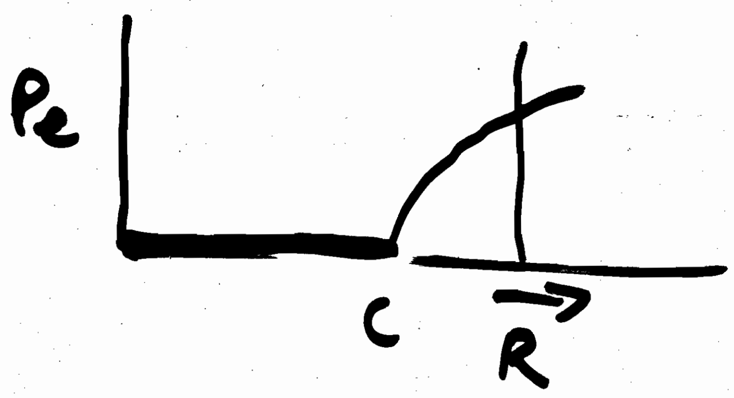
Channel Coding Theorem

$$C \triangleq \max_{p(x)} I(X;Y)$$

For an $\epsilon > 0$ and $R < C$

we can transmit at rate R

with $P_e < \epsilon$.



If $R > C$ then Reliable Communication is not possible

P_e would be more than zero.

$\therefore$ $P_e = 0$ If $R < C$

$P_e > 0$ If $R > C$

Examples Ex-1

Ex $X = \begin{cases} 1 & p \\ 0 & p-1 \end{cases}$ binary r.v.

$$H(X) = p \log \frac{1}{p} + (1-p) \log\left(\frac{1}{(1-p)}\right)$$

$$= -p \log p - (1-p) \log (1-p)$$

$$\triangleq H(p)$$

If $p = 0$

then $H(p) = \{ 0$

$0 \log 0 - (1-0) \log (1-0)$

$0 - 1 \times 0$

0

$1 \log 1 + (1-1) \log \left(\frac{1}{1-p}\right)$

$0 - 0 \times \log\left(\frac{1}{0}\right)$

$\underline{\underline{p = 1/2}}$ $\Rightarrow -1/2 \log_2 (1/2) - (1-1/2) \log (1-1/2)$

$= -1/2 (-1) - \left(\frac{1}{2}\right)(-1)$

$= +1/2 + 1/2 =$

$= 1$

$1/2 \log \left(\frac{1}{1/2}\right) + (1-1/2) \log \frac{1}{(1-1/2)}$

$1/2 \log(2) + 1/2 \log(2)$

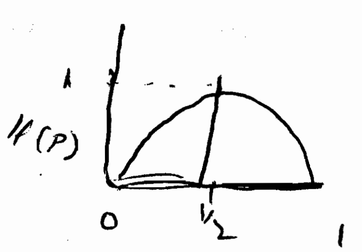
$= 1/2 (1) + 1/2 \times 1$

$=$

If $p = 0$ $H(p) = 0$

$= 1$ $= 0$

$= 1/2$ $= 1$



Ex. a r.v. X

Ex-2

$$\begin{cases} a & 1/2 \\ b & 1/4 \\ c & 1/8 \\ d & 1/8 \end{cases}$$

$$H(X) = \tfrac{1}{2}\log 2 + \tfrac{1}{4}\log 4 + \tfrac{1}{8}\log 8 + \tfrac{1}{8}\log 8$$
$$= 1/2 + 2/4 + 3/8 + 3/8$$
$$= 7/4 \text{ bits} \longrightarrow 1.75$$

If r.v. X → equall probable (4 outcomes)
↳ uniform.

Then $4 \times 1/4 \log_2 4$.

$1 \times 2 = 2$ bits.

a	1/2	$\longrightarrow \log \frac{1}{1/2} \rightarrow 1$	→ 1 bit	0
b	1/4	$\longrightarrow \log \frac{1}{1/4} \rightarrow 2$	→ 2 bits	10
c	1/8		→ 3 bits	110
d	1/8		→ 3 bits	110

Average Code = 7/4 bits → 1.75 bits

Ex

$X = \cancel{Y} \Rightarrow H(X,Y) = H(X) = H(X)$

$x \in \{0, 1\}$ with $p, 1-p$ $\qquad y \in \{0, 1\}$ with $q, 1-q$

Joint Entropy of X & Y

X \ Y	0	1
0	pq	$p(1-q)$
1	$q(1-p)$	$(1-p)(1-q)$

$$H(X;Y) = pq \log \frac{1}{pq} + p(1-q)\log\frac{1}{p(1-q)} + q(1-p)\log\left(\frac{1}{q(1-p)}\right) + (1-p)(1-q)\log\left(\frac{1}{(1-p)(1-q)}\right)$$

$H(p) + H(q)$
$H(X) + H(Y)$

Ex-3

Ex: $X = Y$ $\quad H(X,Y) = H(X) = H(Y)$

$X \in \{0, 1\}$ $\quad Y \in \{0, 1\}$

$\quad p,\ p-1 \qquad\qquad q,\ 1-q$

X \ Y	0	1
0	pq	$p(1-q)$
1	$q(1-p)$	$(1-p)(1-q)$

Joint Entropy

Joint Entropy

$H(X,Y)$

$$= pq \log \frac{1}{pq} + p(1-q) \log \frac{1}{p(1-q)} + q(1-p) \log \frac{1}{q(1-p)} + (1-p)(1-q) \log \frac{1}{(1-p)(1-q)}$$

$$H(p) + H(q)$$

$$= H(X) + H(Y)$$

Ex-4 (9)

Two r.v. X & Y takes 4 values each.

Joint p.d.f.

Y \ X	1	2	3	4
1	1/8	1/16	1/32	1/32
2	1/16	1/8	1/32	1/32
3	1/16	1/16	1/16	1/16
4	1/4	0	0	0

Marginal p.d.f.

$p(X) \rightarrow (1/8+1/16+1/16+1/4,\ 1/16+1/8+1/16+0,\ 1/32+1/32+1/16+0,\ 1/32+1/32+1/16+0)$

$\rightarrow (1/2, 1/4, 1/8, 1/8) \longrightarrow \frac{1}{2}\log 2 + \frac{1}{4}\log 4 + \frac{1}{8}\log 8 + \frac{1}{8}\log 8$

$\Rightarrow 7/4$ bits $\longrightarrow H(X)$

$p(Y) \rightarrow (1/8+1/16+1/32+1/32,\ 1/16+1/8+1/32+1/32,\ 1/16+1/16+1/16+1/16,\ 1/4+0+0+0)$

$(1/4, 1/4, 1/4, 1/4)$

$4\times(\frac{1}{4}\log 1/4) = 1\times 2 = 2$ bits $\rightarrow H(Y)$

$$H(X,Y) = \sum_{x,y} p(x,y)\log\frac{1}{p(x,y)}$$

$= 2\times\frac{1}{8}\log 8 + 6\times\frac{1}{16}\log 16 + 4\times\frac{1}{32}\log 32 + \frac{1}{4}\times\log 4.$

$H(X,Y) = 27/8$ bits.

$$H(X/Y) = \sum_{i=1}^{4} p(Y=i)\, H(X/Y=i)$$

divide by 1/4

Joint dist / Marginal 1/4

$= \frac{1}{4}H(4\times(1/8, 1/16, 1/32, 1/32)) = \frac{1}{4}\overbrace{H(1/2, 1/4, 1/8, 1/8)}^{7.4 \text{ bits}}$

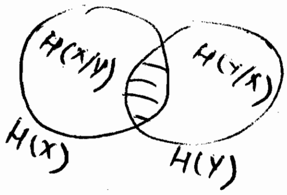
$+ \frac{1}{4}H(4\times(1/16, 1/8, 1/32, 1/32)) + \frac{1}{4}H \ldots$

$+ \frac{1}{4}H(4\times(1/16, 1/16, 1/16, 1/16)) + \frac{1}{4}H(1/4, 1/4, 1/4, 1/4)$

$+ \frac{1}{4}H(4\times(1, 0, 0, 0)) + \frac{1}{4}\ldots$

$\Rightarrow \frac{1}{4}\times 7.4 + \frac{1}{4}\times 7.4 + \frac{1}{4}\times 2 + \frac{1}{4}\times 0$

$\Rightarrow 11/8$



$H(X,Y) = H(X) + H(Y/X)$

$27/8 = 2 + 11/8$

$H(X,Y) = H(X) + H(Y/X)$

$27/8 - 7/4 = H(Y/X) \rightarrow 13/8.$