Illustrative Problems

Chapter 5

Ref: DCS by J.G.Proakis

Digital Communication Systems
MEEIT

T.J.S. Khanzada

HF-IESK

OTTO VON GUERICKE UNIVERSITÄT MAGDEBURG

FAKULTÄT FÜR ELEKTROTECHNIK UND INFORMATIONSTECHNIK

ILLUSTRATIVE PROBLEM

Illustrative Problem 5.1 Suppose the signal waveforms $s_0(t)$ and $s_1(t)$ are the ones shown in Figure 5.2, and let $s_0(t)$ be the transmitted signal. Then, the received signal is

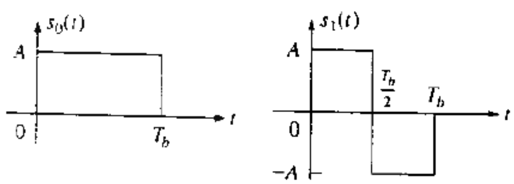


Figure 5.2: Signal waveforms $s_0(t)$ and $s_1(t)$ for a binary communication system.

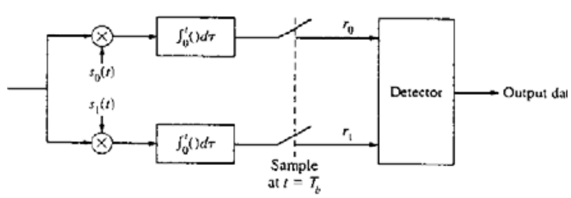


Figure 5.1: Cross correlation of the received signal $r(t)$ with the two transmitted signals.

$$r(t) = s_0(t) + n(t), \qquad 0 \le t \le T_b \tag{5.2.3}$$

Determine the correlator outputs at the sampling instants.

SOLUTION

When the signal $r(t)$ is processed by the two signal correlators shown in Figure 5.1, the outputs r_0 and r_1 at the sampling instant $t = T_b$ are

$$\begin{aligned} r_0 &= \int_0^{T_b} r(t) s_0(t)\, dt \\ &= \int_0^{T_b} s_0^2(t)\, dt + \int_0^{T_b} n(t) s_0(t)\, dt \\ &= \mathcal{E} + n_0 \end{aligned}$$

and
$$r_1 = \int_0^{T_b} r(t)s_1(t)\,dt$$
$$= \int_0^{T_b} s_0(t)s_1(t)\,dt + \int_0^{T_b} n(t)s_1(t)\,dt$$
$$= n_1$$

where n_0 and n_1 are the noise components at the output of the signal correlators, i.e.,

$$n_0 = \int_0^{T_b} n(t)s_0(t)\,dt$$
$$n_1 = \int_0^{T_b} n(t)s_1(t)\,dt$$

and $\mathcal{E} = A^2 T_b$ is the energy of the signals $s_0(t)$ and $s_1(t)$. We also note that the two signal waveforms are *orthogonal*, i.e.,

$$\int_0^{T_b} s_0(t)s_1(t)\,dt = 0$$

On the other hand, when $s_1(t)$ is the transmitted signal, the received signal is

$$r(t) = s_1(t) + n(t), \qquad 0 \le t \le T_b$$

It is easy to show that, in this case, the signal correlator outputs are

$$r_0 = n_0$$
$$r_1 = \mathcal{E} + n_1$$

Figure 5.3 illustrates the two noise-free correlator outputs in the interval $0 \le t \le T_b$ for each of the two cases—i.e., when $s_0(t)$ is transmitted and when $s_1(t)$ is transmitted.

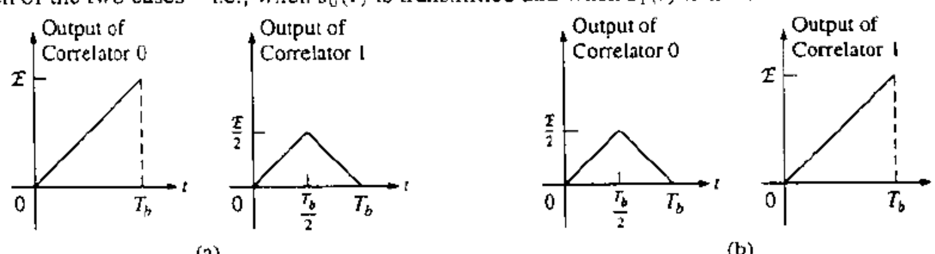


Figure 5.3: Noise-free correlator outputs. (a) $s_0(t)$ was transmitted. (b) $s_1(t)$ was transmitted.

Since $n(t)$ is a sample function of a white Gaussian process with power spectrum $N_0/2$ the noise components n_0 and n_1 are Gaussian with zero means, i.e.,

$$E(n_0) = \int_0^{T_b} s_0(t)E[n(t)]\,dt = 0$$
$$E(n_1) = \int_0^{T_b} s_1(t)E[n(t)]\,dt = 0$$

and variances σ_i^2, for $i = 1, 2$, where

$$\sigma_i^2 = E(n_i^2)$$
$$= \int_0^{T_b}\int_0^{T_b} s_i(t)s_i(\tau)E[n(t)n(\tau)]\,dt\,d\tau$$
$$= \frac{N_0}{2}\int_0^{T_b} s_i(t)s_i(\tau)\delta(t-\tau)\,dt\,d\tau$$
$$= \frac{N_0}{2}\int_0^{T_b} s_i^2(t)\,dt$$
$$= \frac{\mathcal{E}N_0}{2}, \qquad i = 0, 1$$

Therefore, when $s_0(t)$ is transmitted, the probability density functions of r_0 and r_1 are

$$p(r_0 \mid s_0(t) \text{ was transmitted}) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(r_0-\mathcal{E})^2/2\sigma^2}$$
$$p(r_1 \mid s_0(t) \text{ was transmitted}) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-r_1^2/2\sigma^2}$$

These two probability density functions, denoted as $p(r_0 \mid 0)$ and $p(r_1 \mid 0)$, are illustrated in Figure 5.4. Similarly, when $s_1(t)$ is transmitted, r_0 is zero-mean Gaussian with variance σ^2 and r_1 is Gaussian with mean value $\mathcal{E}$ and variance σ^2.

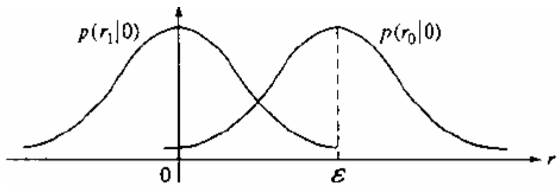


Figure 5.4: Probability density function $p(r_0 \mid 0)$ and $p(r_1 \mid 0)$ when $s_0(t)$ is transmitted.

ILLUSTRATIVE PROBLEM

Illustrative Problem 5.2 Consider the use of matched filters for the demodulation of the two signal waveforms shown in Figure 5.2 and determine the outputs.

SOLUTION

The impulse responses of the two matched filters are

$$h_0(t) = s_0(T_b - t)$$
$$h_1(t) = s_1(T_b - t)$$

as illustrated in Figure 5.5. Note that each impulse response is obtained by folding the signal $s(t)$ to obtain $s(-t)$ and then delaying the folded signal $s(-t)$ by T_b to obtain $s(T_b - t)$.

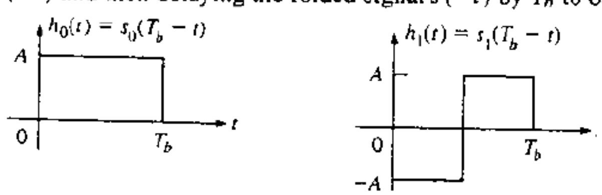


Figure 5.5: Impulse responses of matched filters for signals $s_0(t)$ and $s_1(t)$

Now suppose the signal waveforms $s_0(t)$ is transmitted. Then, the received signal $r(t) = s_0(t) + n(t)$ is passed through the two matched filters. The response of the filter with impulse response $h_0(t)$ to the signal component $s_0(t)$ is illustrated in Figure 5.6(a). Also, the response of the filter with impulse response $h_1(t)$ to the signal component $s_0(t)$ is illustrated in Figure 5.6(b). Hence, at the sampling instant $t = T_b$, the outputs of the two matched filters with impulse responses $h_0(t)$ and $h_1(t)$ are

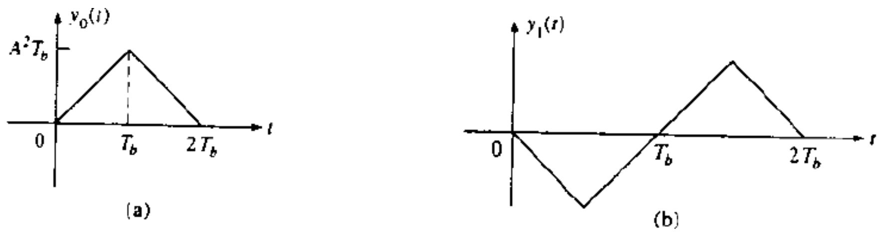


Figure 5.6: Signal outputs of matched filters when $s_0(t)$ is transmitted.

$$r_0 = \mathcal{E} + n_0$$
$$r_1 = n_1$$

respectively. Note that these outputs are identical to the outputs obtained from sampling the signal correlator outputs at $t = T_b$.

ILLUSTRATIVE PROBLEM

Illustrative Problem 5.3 Let us consider the detector for the signals shown in Figure 5.2 which are equally probable and have equal energies. The optimum detector for these signals compares r_0 and r_1 and decides that a 0 was transmitted when $r_0 > r_1$ and that a 1 was transmitted when $r_1 > r_0$. Determine the probability of error.

SOLUTION

When $s_0(t)$ is the transmitted signal waveform, the probability of error is

$$P_e = P(r_1 > r_0) = P(n_1 > \mathcal{E} + n_0) = P(n_1 - n_0 > \mathcal{E})$$

Since n_1 and n_0 are zero-mean Gaussian random variables, their difference $x \equiv n_1 - n_0$ is also zero-mean Gaussian. The variance of the random variable x is

$$E(x^2) = E[(n_1 - n_0)^2] = E(n_1^2) + E(n_0^2) - 2E(n_1 n_0)$$

But $E(n_1 n_0) = 0$, because the signal waveforms are orthogonal. That is

$$\begin{aligned} E(n_1 n_0) &= E \int_0^{T_b} \int_0^{T_b} s_0(t) s_1(\tau) n(t) n(\tau)\, dt\, d\tau \\ &= \frac{N_0}{2} \int_0^{T_b} \int_0^{T_b} s_0(t) s_1(\tau) \delta(t - \tau)\, dt\, d\tau \\ &= \frac{N_0}{2} \int_0^{T_b} s_0(t) s_1(t)\, dt \\ &= 0 \end{aligned}$$

Therefore,

$$E(x^2) = 2\left(\frac{\mathcal{E} N_0}{2}\right) = \mathcal{E} N_0 \equiv \sigma_x^2$$

Hence, the probability of error is

$$
\begin{aligned}
P_e &= \frac{1}{\sqrt{2\pi}\,\sigma_x}\int_{\mathcal{E}}^{\infty} e^{-x^2/2\sigma_x^2}\,dx \\
&= \frac{1}{\sqrt{2\pi}}\int_{\sqrt{\mathcal{E}/N_0}}^{\infty} e^{-x^2/2}\,dx \\
&= Q\left(\sqrt{\frac{\mathcal{E}}{N_0}}\right) \quad \text{Eq.C}
\end{aligned}
$$

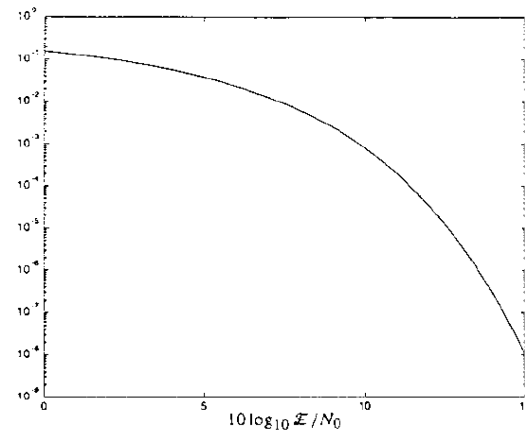


Figure 5.7: Probability of error for orthogonal signals

The ratio $\mathcal{E}/N_0$ is called the signal-to-noise ratio (SNR)

The derivation of the detector performance given above was based on the transmission of the signal waveform $s_0(t)$. The reader may verify that the probability of error that is obtained when $s_1(t)$ is transmitted is identical to that obtained when $s_0(t)$ is transmitted. Because the 0's and 1's in the data sequence are equally probable, the average probability of error is that given by Eq.C This expression for the probability of error is evaluated and is plotted in Figure 5.7 as a function of th SNF

where the SNR is displayed on a logarithmic scale ($10 \log_{10} \mathcal{E}/N_0$)

As expected, the probability of error decreases exponentially as the SNR increases

ILLUSTRATIVE PROBLEM

Illustrative Problem 5.4 Use Monte Carlo simulation to estimate and plot P_e versus SNR for a binary communication system that employs correlators or matched filters. The model of the system is illustrated in Figure 5.8.

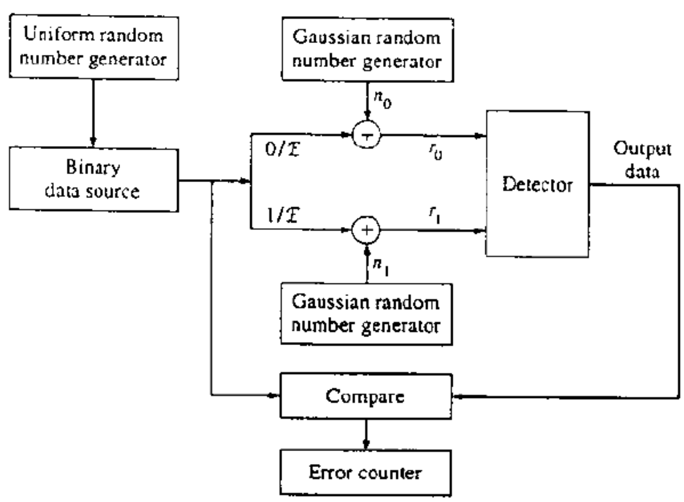


Figure 5.8: Simulation model for Illustrative Problem 5.4

SOLUTION

We simulate the generation of the random variables r_0 and r_1, which constitute the input to the detector. We begin by generating a binary sequence of 0's and 1's that occur with equal probability and are mutually statistically independent.
To accomplish this task, we use a random number generator

that generates a uniform random number with a range (0, 1).

If the number generated is in the range (0, 0.5),

the binary source output is a 0.

Otherwise,

it is a 1

If a 0 is generated,

then $r_0 = \mathcal{E} + n_0$ and $r_1 = n_1$

If a 1 is generated

then $r_0 = n_0$ and $r_1 = \mathcal{E} + n_1$

The additive noise components n_0 and n_1 are generated by means of two Gaussian noise generators. Their means are zero and their variances are $\sigma^2 = \mathcal{E} N_0/2$. For convenience, we may normalize the signal energy $\mathcal{E}$ to unity ($\mathcal{E} = 1$) and vary σ^2. Note that the SNR, which is defined as $\mathcal{E}/N_0$, is then equal to $1/2\sigma^2$. The detector output is compared with the binary transmitted sequence, and an error counter is used to count the number of bit errors.

Figure 5.9 illustrates the results of this simulation for the transmission of N=10,000 bits at several different values of SNR. Note the agreement between the simulation results and the theoretical value of P_e given by Eq.C. We should also note that a simulation of N=10,000 data bits allows us to estimate the error probability reliably down to about $P_e = 10^{-3}$. In other words, with N=10,000 data bits, we should have at least 10 errors for a reliable estimate of P_e.

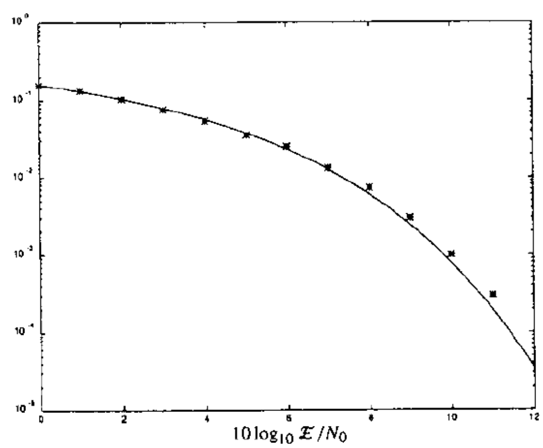


Figure 5.9: Error probability from Monte Carlo simulation compared with theoretical error probability for orthogonal signaling.

QUESTION

In Figure 5.9, simulation and theoretical results completely agree at low signal-to-noise ratios, whereas at higher SNRs they agree less. Can you explain why? How should we change the simulation process to result in better agreement at higher signal-to-noise ratios?

ILLUSTRATIVE PROBLEM

Illustrative Problem 5.5 Use Monte Carlo simulation to estimate and plot the error probability performance of a binary antipodal communication system. The model of the system is illustrated in Figure 5.13.

SOLUTION

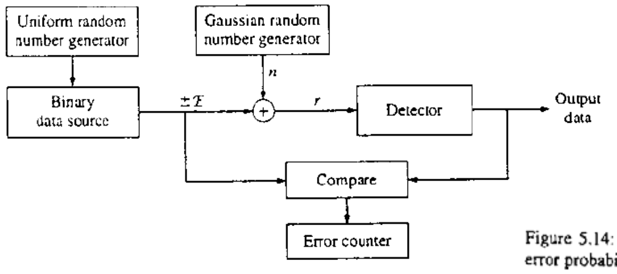


Figure 5.13: Model of the binary communication system employing antipodal signals.

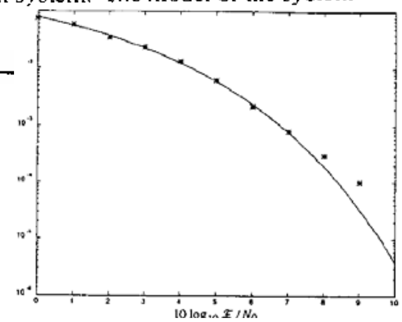


Figure 5.14: Error probability from Monte Carlo simulation compared with theoretical error probability for antipodal signals.

As shown, we simulate the generation of the random variable r, which is the input to the detector. A uniform random number generator is used to generate the binary information sequence from the binary data source. The sequence of 0's and 1's is mapped into a sequence of $\pm\mathcal{E}$, where $\mathcal{E}$ represents the signal energy. $\mathcal{E}$ may be normalized to unity. A Gaussian noise generator is used to generate the sequence of zero-mean Gaussian random numbers with variance σ^2. The detector compares the random variable r with the threshold of 0. If $r > 0$, the decision is made that the transmitted bit is a 0. If $r < 0$, the decision is made that the transmitted bit is a 1. The output of the detector is compared with the transmitted sequence of information bits, and the bit errors are counted. Figure 5.14 illustrates the results of this simulation for the transmission of 10,000 bits at several different values of SNR. The theoretical value for P_e given by Eq.D is also plotted in Figure 5.14 for

$$
\begin{aligned}
P_e &= P(r<0)\\
&= \frac{1}{\sqrt{2\pi}\,\sigma}\int_{-\infty}^{0} e^{-(r-\mathcal{E})^2/2\sigma^2}\,dr\\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-\mathcal{E}/\sigma} e^{-r^2/2}\,dr\\
&= Q\left(\frac{\mathcal{E}}{\sigma}\right)\\
&= Q\left(\sqrt{\frac{2\mathcal{E}}{N_0}}\right) \quad \text{Eq.D}
\end{aligned}
$$

ILLUSTRATIVE PROBLEM

Illustrative Problem 5.6 Use Monte Carlo simulation to estimate and plot the performance of a binary communication system employing on-off signaling.

SOLUTION

The model for the system to be simulated is similar to that shown in Figure 5.13, except that one of the signals is 0. Thus, we generate a sequence of random variables $\{r_i\}$ as given

$$\text{by: } r = \begin{cases} n, & \text{if 0 is transmitted} \\ \mathcal{E} + n, & \text{if a 1 is transmitted} \end{cases}$$

The detector compares the random variables $\{r_i\}$ to the optimum threshold $\mathcal{E}/2$ and makes the appropriate decisions. Figure 5.16 illustrates the estimated error probability based on 10,000 binary digits. The theoretical error rate given by

$$P_e(\alpha_{opt}) = Q\left(\sqrt{\frac{\mathcal{E}}{2N_0}}\right)$$

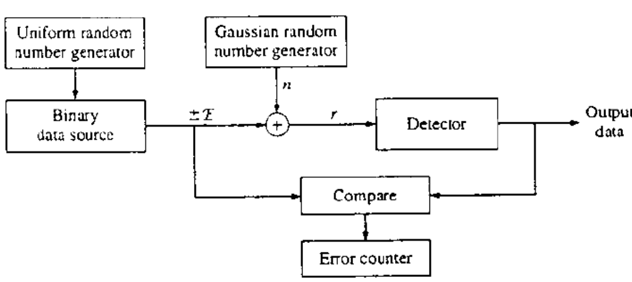


Figure 5.13: Model of the binary communication system employing antipodal signals.

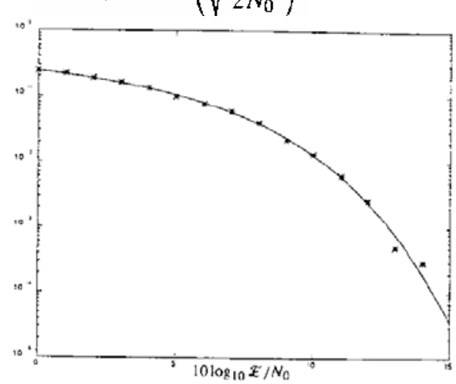


Figure 5.16: Error probability from Monte Carlo simulation compared with theoretical error probability for on-off signals.

ILLUSTRATIVE PROBLEM

Illustrative Problem 5.7 The effect of noise on the performance of a binary communication system can be observed from the received signal plus noise at the input to the detector. For example, let us consider binary orthogonal signals, for which the input to the detector consists of the pair of random variables (r_0, r_1), where either $(r_0, r_1) = (\sqrt{\mathcal{E}} + n_0, n_1)$

or $\quad (r_0, r_1) = (n_0, \sqrt{\mathcal{E}} + n_1)$

The noise random variables n_0 and n_1 are zero-mean, independent Gaussian random variables with variance σ^2. As in Illustrative Problem 5.4, use Monte Carlo simulation to generate 100 samples of (r_0, r_1) for each value of $\sigma = 0.1$, $\sigma = 0.3$, and $\sigma = 0.5$, and plot these 100 samples for each σ on different two-dimensional plots. The energy $\mathcal{E}$ of the signal may be normalized to unity.

SOLUTION

The results of the Monte Carlo simulation are shown in Figure 5.18. Note that at a low noise power level (σ small) the effect of the noise on performance (error rate) of the communication system is small. As the noise power level increases, the noise components increase in size and cause more errors.

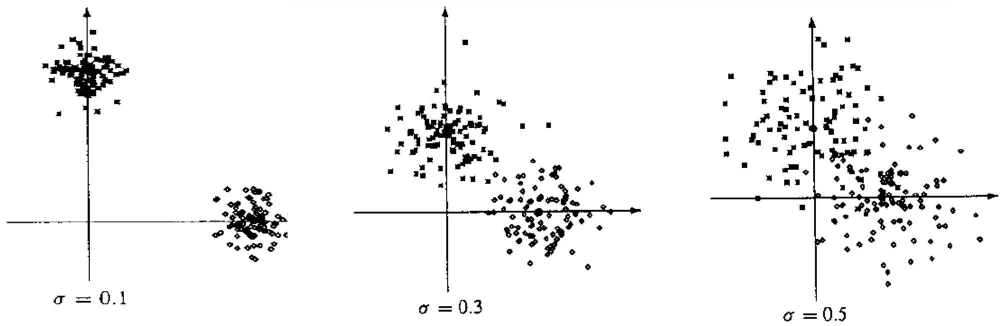


Figure 5.18: Received signal points at input to the selector for orthogonal signals (Monte Carlo simulation).

ILLUSTRATIVE PROBLEM

Illustrative Problem 5.8 Perform a Monte Carlo simulation of the four-level (quaternary) PAM communication system that employs a signal correlator, as described above, followed by an amplitude detector. The model for the system to be simulated is shown in Figure 5.22.

SOLUTION

As shown, we simulate the generation of the random variable r, which is the output of the signal correlator and the input to the detector. We begin by generating a sequence of quaternary symbols that are mapped into corresponding amplitude levels $\{A_m\}$. To accomplish this task, we use a random number generator that generates a uniform random number in the range (0, 1). This range is subdivided into four equal intervals, (0, 0.25), (0.25, 0.5), (0.5, 0.75), (0.75, 1.0), where the subintervals correspond to the symbols (pairs of information bits) 00, 01, 11, 10, respectively. Thus, the output of the uniform random number generator is mapped into the corresponding signal amplitude levels $(-3d, -d, d, 3d)$, respectively.

The additive noise component having mean 0 and variance σ^2 is generated by means of a Gaussian random number generator. For convenience, we may normalize the distance parameter $d = 1$ and vary σ^2. The detector observes $r = A_m + n$ and computes the distance between r and the four possible transmitted signal amplitudes. Its output $\hat{A}_m$ is the signal amplitude level corresponding to the smallest distance. $\hat{A}_m$ is compared with the actual transmitted signal amplitude, and an error counter is used to count the errors made by the detector.

Figure 5.23 illustrates the results of the simulation for the transmissions of N=10,000 symbols at different values of the average bit SNR, which is defined as

$$\frac{\mathcal{E}_{avb}}{N_0} = \frac{5}{4}\left(\frac{d^2}{\sigma^2}\right)$$

Note the agreement between the simulation results and the theoretical values of P_4 computed

$$P_4 = \frac{3}{2} Q\left(\sqrt{\frac{2\mathcal{E}_{av}}{5N_0}}\right)$$

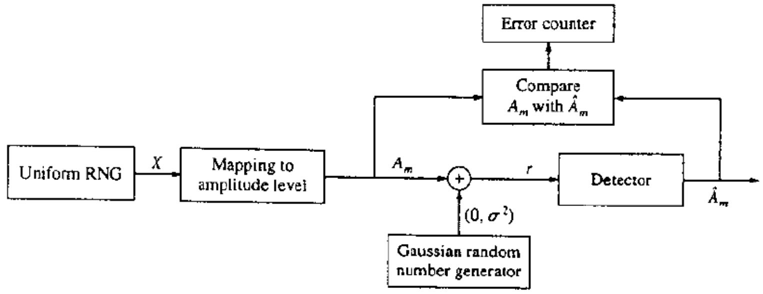


Figure 5.22: Block diagram of four-level PAM for Monte Carlo simulation

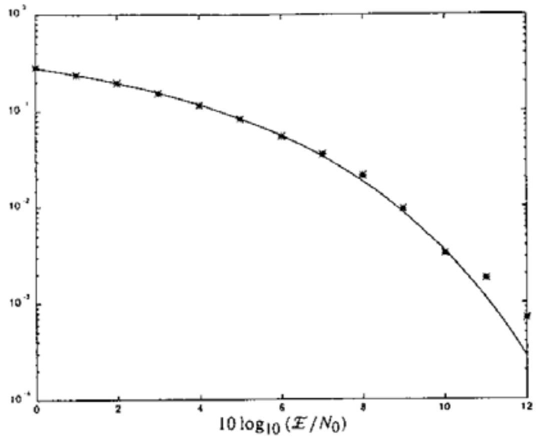


Figure 5.23: Error probability for Monte Carlo simulation compared with theoretical error probability for $M = 4$ PAM.

ILLUSTRATIVE PROBLEM

Illustrative Problem 5.10 Perform a Monte Carlo simulation of a digital communication system that employs $M = 4$ orthogonal signals. The model of the system to be simulated is illustrated in Figure 5.30.

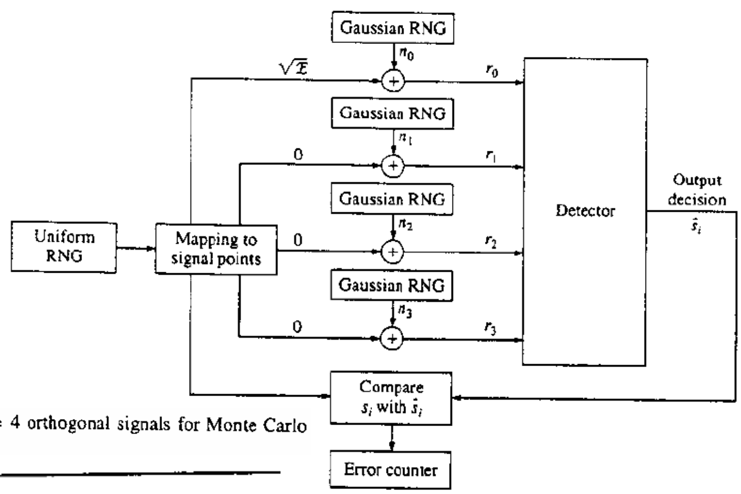


Figure 5.30: Block diagram of system with $M = 4$ orthogonal signals for Monte Carlo simulation.

SOLUTION

As shown, we simulate the generation of the random variables r_0, r_1, r_2, r_3, which constitute the input to the detector. We may first generate a binary sequence of 0's and 1's that occur with equal probability and are mutually statistically independent, as in Illustrative Problem 5.4. The binary sequence is grouped into pairs of bits, which are mapped into the corresponding signal components. An alternative to generating the individual bits is to generate the pairs of bits, as in Illustrative Problem 5.8. In any case, we have the mapping of the four symbols into the signal points:

$$00 \rightarrow s_0 = (\sqrt{\mathcal{E}}, 0, 0, 0)$$
$$01 \rightarrow s_1 = (0, \sqrt{\mathcal{E}}, 0, 0)$$
$$10 \rightarrow s_2 = (0, 0, \sqrt{\mathcal{E}}, 0)$$
$$11 \rightarrow s_3 = (0, 0, 0, \sqrt{\mathcal{E}})$$

The additive noise components n_0, n_1, n_2, n_3 are generated by means of four Gaussian noise generators, each having a mean zero and variance $\sigma^2 = N_0\mathcal{E}/2$. For convenience, we may normalize the symbol energy to $\mathcal{E} = 1$ and vary σ^2. Since $\mathcal{E} = 2\mathcal{E}_b$, it follows that $\mathcal{E}_b = \frac{1}{2}$. The detector output is compared with the transmitted sequence of bits, and an error counter is used to count the number of bit errors.

Figure 5.31 illustrates the results of this simulation for the transmission of 20,000 bits at several different values of the SNR $\mathcal{E}_b/N_0$. Note the agreement between the simulation results and the theoretical value of P_b given by

$$P_b = \frac{2^{k-1}}{2^k - 1} P_M$$

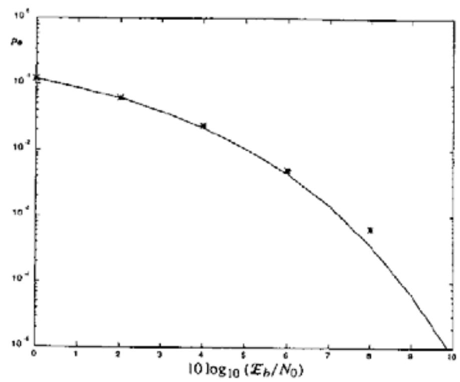


Figure 5.31: Bit error probability for $M = 4$ orthogonal signals from a Monte Carlo simulation compared with theoretical error probability.

ILLUSTRATIVE PROBLEM

Illustrative Problem 5.9 Perform a Monte Carlo simulation of a 16-level PAM digital communication system and measure its error-rate performance.

SOLUTION

The basic block diagram shown in Figure 5.22 applies in general. A uniform random number generator is used to generate the sequence of information symbols, which are viewed as blocks of four information bits. The 16-ary symbols may be generated directly by subdividing the interval (0, 1) into 16 equal-width subintervals and mapping the 16-ary symbols into the 16-ary signal amplitudes. A white Gaussian noise sequence is added to the 16-ary information symbol sequence, and the resulting signal plus noise is fed to the detector. The detector computes the distance metrics given by Eq.I) and selects the

$$D_i = |r - A_i|, \qquad i = 0, 1, 2, 3 \qquad \text{Eq.I}$$

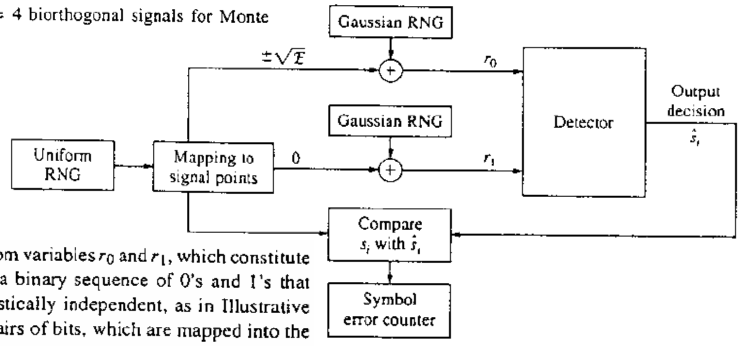
amplitude corresponding to the smallest metric. The output of the detector is compared with the transmitted information symbol sequence and errors and counted. Figure 5.25 illustrates the measured symbol error rate for 10,000 transmitted symbols and the theoretical symbol error rate given by Eq.J with $M = 16$.

$$P_M = \frac{2(M-1)}{M} Q\left(\sqrt{\frac{6(\log_2 M)\mathcal{E}_{avb}}{(M^2-1)N_0}}\right) \qquad \text{Eq.J}$$

ILLUSTRATIVE PROBLEM

Illustrative Problem 5.11 Perform a Monte Carlo simulation for a digital communication system that employs $M = 4$ biorthogonal signals. The model of the system to be simulated is illustrated in Figure 5.34.

Figure 5.34: Block diagram of the system with $M = 4$ biorthogonal signals for Monte Carlo simulation.



SOLUTION

As shown, we simulate the generation of the random variables r_0 and r_1, which constitute the input to the detector. We begin by generating a binary sequence of 0's and 1's that occur with equal probability and are mutually statistically independent, as in Illustrative Problem 5.4. The binary sequence is grouped into pairs of bits, which are mapped into the corresponding signal components as follows:

$$00 \rightarrow s_0 = (\sqrt{\mathcal{E}}, 0)$$
$$01 \rightarrow s_1 = (0, \sqrt{\mathcal{E}})$$
$$10 \rightarrow s_2 = (0, -\sqrt{\mathcal{E}})$$
$$11 \rightarrow s_3 = (-\sqrt{\mathcal{E}}, 0)$$

Alternatively, we may use the method in Illustrative Problem 5.8 to generate the 2-bit symbols directly.

Since $s_2 = -s_1$ and $s_3 = -s_0$, the demodulation requires only two correlators or matched filters, whose outputs are r_0 and r_1. The additive noise components n_0 and n_1 are generated by means of two Gaussian noise generators, each having a mean zero and variance $\sigma^2 = N_0\mathcal{E}/2$. For convenience, we may normalize the symbol energy to $\mathcal{E} = 1$ and vary σ^2. Since $\mathcal{E} = 2\mathcal{E}_b$, it follows that $\mathcal{E}_b = \frac{1}{2}$. The detector output is compared with the transmitted sequence of bits, and an error counter is used to count the number of symbol errors and the number of bit errors.

Figure 5.35 illustrates the results of this simulation for the transmission of 20,000 bits at several different values of the SNR $\mathcal{E}_b/N_0$. Note the agreement between the simulation results and the theoretical value of P_4 given by

$$P_c = \int_0^{\infty} \left[\frac{1}{\sqrt{2\pi}} \int_{-r_0/\sqrt{\mathcal{E}N_0/2}}^{r_0/\sqrt{\mathcal{E}N_0/2}} e^{-x^2/2}\, dx \right]^{M-1} p(r_0)\, dr_0$$

and

$$P_M = 1 - P_c$$