Power and Energy of a Signal

(Reference Material for MEIT-DCS-lect2, prepared by Khanzada)

$$E_X = \int_{-\infty}^{\infty} x^2(t)\,dt$$

Energy of a real signal

$$P_X = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} x^2(t)\,dt$$

Power of a real signal

Energy-type Signal

- A signal with finite energy is called an *energy-type signal*
- The *energy spectral density* of an energy-type signal gives the distribution of energy at various frequencies of the signal

$$\mathcal{G}_X(f) = |X(f)|^2$$

$$E_X = \int_{-\infty}^{\infty} \mathcal{G}_X(f)\,df$$

Example of energy type signal $x(t) = \Pi(t)$

Power-type Signal

- A signal with finite power is a *power-type signal.*
- All periodic Signals are Power-Type Signals except those which are zero almost everywhere.

Example of energy type signal $x(t) = \cos(t)$

Some signals are neither energy nor power type e.g. $e^t u_{-1}(t)$.

- The **Auto-Correlation function** of x(t) (real valued signal) is $R_X(\tau)$

$$R_X(\tau) = \int_{-\infty}^{\infty} x(t)x(t+\tau)\,dt$$
$$= x(\tau) \star x(-\tau)$$

- The **Time-Average Auto-Correlation function** (for power signals)

$$R_X(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)x(t+\tau)\,dt$$

- The **Power Spectral Density (P.S.D)** for a signal is

$$S_X(f) = \mathcal{F}[R_X(\tau)]$$

- The **Total Power** id the integral of the P.S.D. is

$$P_X = \int_{-\infty}^{\infty} S_X(f)\,df$$

- The **F.S. coefficients x_n of the P.S.D.** is (for periodic signal x(t), Period=T_0)

$$S_X(f) = \sum_{n=-\infty}^{\infty} |x_n|^2 \delta\left(f - \frac{n}{T_0}\right)$$

Energy S.D. and Power S.D. (for x(t) filtered by H(f))

$$\mathcal{G}_Y(f) = |H(f)|^2 \mathcal{G}_X(f)$$
$$S_Y(f) = |H(f)|^2 S_X(f)$$

- Energy and Power of the **discrete time signals** are

(infinite sequence)

$$E_X = T_s \sum_{n=-\infty}^{\infty} x^2[n]$$

$$P_X = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} x^2[n]$$

(finite sequence)

$$E_X = T_s \sum_{n=0}^{N-1} x^2[n]$$

$$P_X = \frac{1}{N} \sum_{n=0}^{N-1} x^2[n]$$

SOME FOURIER TRANSFORM PAIRS

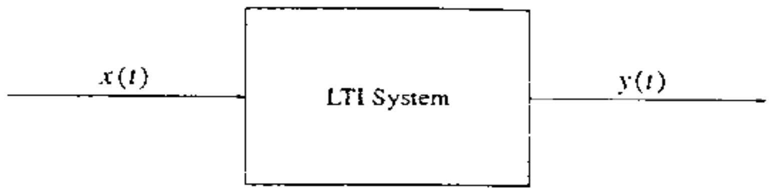
$x(t)$	$X(f)$
$\delta(t)$	1
1	$\delta(f)$
$\delta(t-t_0)$	$e^{-j2\pi f t_0}$
$e^{j2\pi f t_0}$	$\delta(f-f_0)$
$\cos(2\pi f_0 t)$	$\frac{1}{2}\delta(f-f_0)+\frac{1}{2}\delta(f+f_0)$
$\sin(2\pi f_0 t)$	$\frac{1}{2j}\delta(f-f_0)-\frac{1}{2j}\delta(f+f_0)$
$\Pi(t)$	$\mathrm{sinc}(f)$
$\mathrm{sinc}(t)$	$\Pi(f)$
$\Lambda(t)$	$\mathrm{sinc}^2(f)$
$\mathrm{sinc}^2(t)$	$\Lambda(f)$
$e^{-\alpha t}u_{-1}(t), \quad \alpha>0$	$\frac{1}{\alpha+j2\pi f}$
$te^{-\alpha t}u_{-1}(t), \quad \alpha>0$	$\frac{1}{(\alpha+j2\pi f)^2}$
$e^{-\alpha\|t\|}, \quad \alpha>0$	$\frac{2\alpha}{\alpha^2+(2\pi f)^2}$
$e^{-\pi t^2}$	$e^{-\pi f^2}$
$\mathrm{sgn}(t)$	$\frac{1}{j\pi f}$
$u_{-1}(t)$	$\frac{1}{2}\delta(f)+\frac{1}{j2\pi f}$
$\delta'(t)$	$j2\pi f$
$\delta^{(n)}(t)$	$(j2\pi f)^n$
$\sum_{n=-\infty}^{\infty}\delta(t-nT_0)$	$\frac{1}{T_0}\sum_{n=-\infty}^{\infty}\delta\left(f-\frac{n}{T_0}\right)$

SOME FOURIER TRANSFORM PAIRS

Periodic Signal passed through Linear Time Invariant (LTI) system

(Reference Material for MEIT-DCS-lect2, prepared by Khanzada)

Let x(t) be a periodic signal passed though the Linear Time Invariant (LTI) system to get y(t) output signal



where $x(t) = \sum_{n=-\infty}^{\infty} x_n e^{j2\pi nt/T_0}$ and $y(t) = \sum_{n=-\infty}^{\infty} y_n e^{j2\pi nt/T_0}$

then the relationship between the Fourier series coefficients of x(t) and y(t) can obtained by employing the convolution integral as under

$$
\begin{aligned}
y(t) &= \int_{-\infty}^{\infty} x(t-\tau)h(\tau)\,d\tau \\
&= \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x_n e^{j2\pi n(t-\tau)/T_0} h(\tau)\,d\tau \\
&= \sum_{n=-\infty}^{\infty} x_n \left(\int_{-\infty}^{\infty} h(\tau) e^{-j2\pi n\tau/T_0}\,d\tau \right) e^{j2\pi nt/T_0} \\
&= \sum_{n=-\infty}^{\infty} y_n e^{j2\pi nt/T_0}
\end{aligned}
$$

$$\Rightarrow \quad y_n = x_n H\left(\frac{n}{T_0}\right)$$

where ***H(f)*** is the transfer function of the LTI system achieved by applying the F.T. on its Impulse Response (I.R.) ***h(t)***

$$H(f) = \int_{-\infty}^{\infty} h(t) e^{-j2\pi ft}\,dt$$

The output of an LTI system with impulse response $h(t)$ when the input signal is $x(t)$ is given by the convolution integral

$$y(t) = x(t) \star h(t)$$

Applying the convolution theorem we obtain

$$Y(f) = X(f)H(f)$$

where

$$H(f) = \mathcal{F}[h(t)] = \int_{-\infty}^{\infty} h(t)e^{-j2\pi ft}\,dt$$

is the transfer function of the system.

<u>Magnitude and Phase spectra</u> relationship of the above equations can be rewritten as

$$|Y(f)| = |X(f)|\,|H(f)|$$

$$\angle Y(f) = \angle X(f) + \angle H(f)$$

respectively

Lowpasss equivalent of Bandpass Signals

- Bandpass Signal

A bandpass signal is a signal for which all frequency components are located in the neighborhood of a *central frequency* f_0 **and $-f_0$**

It can be represented as $X(f) \equiv 0$ for $|f \pm f_0| > W$, where $W \ll f_0$.

- Lowpass Signal

A lowpass signal is a signal for which the frequency components are located around the zero frequency i.e. for $|f| > W$, $X(f) \equiv 0$.

- Analytic Signal z(t) **(corresponding to bandpass signal x(t))**

$z(t) = x(t) + j\hat{x}(t)$ and its F.T. $Z(f) = 2u_{-1}(f)X(f)$

where $u_{-1}(f)$ is the unit step function.
$\hat{x}(t)$ denotes the *Hilbert transform* of $x(t)$

- Hilbert Transform $\hat{x}(t)$ of x(t)

Hilbert Transform is defined as $\hat{x}(t) = x(t) \star \frac{1}{\pi t}$; in time domain and as

$\hat{X}(f) = -j\,\mathrm{sgn}(f)X(f)$ in frequency domain

The *lowpass equivalent* of the signal $x(t)$, denoted by $x_l(t)$, is expressed in terms of $z(t)$ as

$$x_l(t) = z(t)e^{-j2\pi f_0 t}$$

we can derive now

$$x(t) = \mathrm{Re}[x_l(t)e^{j2\pi f_0 t}]$$

$$\hat{x}(t) = \mathrm{Im}[x_l(t)e^{j2\pi f_0 t}]$$

in time domain and

$$X_l(f) = Z(f + f_0) = 2u_{-1}(f + f_0)X(f + f_0)$$ and

$$X_l(f) = X(f - f_0) + X^*(-f - f_0)$$

- **In-Phase** and **Quadrature** components of a Complex Signal

The lowpass equivalent of a real bandpass signal is, in general, a complex signal. Its real part, denoted by $x_c(t)$, is called the *in-phase component* of $x(t)$ and its imaginary part is called the *quadrature component* of $x(t)$ and is denoted by $x_s(t)$; i.e.,

$$x_l(t) = x_c(t) + jx_s(t)$$

In terms of the in-phase and the quadrature components we have

$$\begin{cases} x(t) = x_c(t)\cos(2\pi f_0 t) - x_s(t)\sin(2\pi f_0 t) \\ \hat{x}(t) = x_s(t)\cos(2\pi f_0 t) + x_c(t)\sin(2\pi f_0 t) \end{cases}$$

- **Envelope and Phase of a Signal**

If we express $x_l(t)$ in polar coordinates, we have

$$x_l(t) = V(t)e^{j\Theta(t)}$$

where $V(t)$ and $\Theta(t)$ are called the *envelope* and the *phase* of the signal $x(t)$. In terms of these two we have

$$x(t) = V(t)\cos(2\pi f_0 t + \Theta(t))$$

The envelope and the phase can be expressed as

$$V(t) = \sqrt{x_c^2(t) + x_s^2(t)} \qquad \text{OR} \qquad V(t) = \sqrt{x^2(t) + \hat{x}^2(t)}$$

$$\Theta(t) = \arctan\frac{x_s(t)}{x_c(t)} \qquad\qquad \Theta(t) = \arctan\frac{\hat{x}(t)}{x(t)} - 2\pi f_0 t$$

The envelope is independent of the choice of f_0, but the phase depends on this choice.