Summer University
Speech Recognition
01 July – 30 August 2002

2nd Lecture & Computer Experiments
Digital Signal Processing
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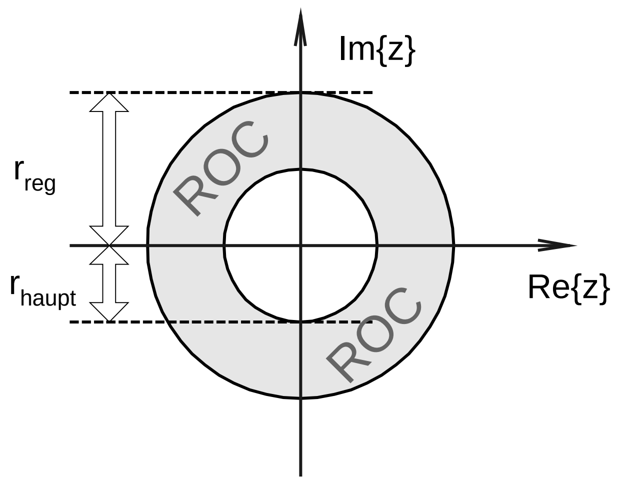
Schedule Lecture 2: Deterministic Systems

- **Refresh**
- **Systems Analysis and Design**
- **System's response in the frequency domain**
- **Discrete /fast Fourier transforms**



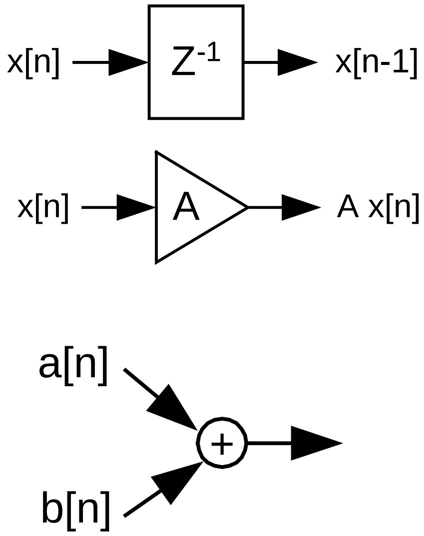
1Direkte Integration

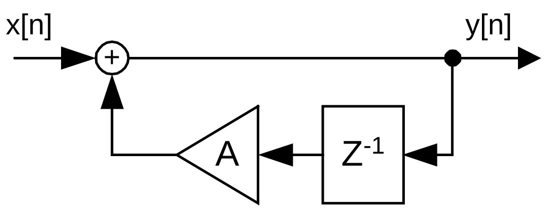
2Anwendung des Residuensatzes

3Zerlegung in Teile, deren Inverse bekannt ist (Korrespondenztabelle)

4Bei algebraischen Funktionen: Partialbruchzerlegung

Difference Equations:





$$y[n] = x[n] + A\,y[n-1]$$

$$y[n] = \frac{A^{n+1}-1}{A-1}\,\sigma[n]$$

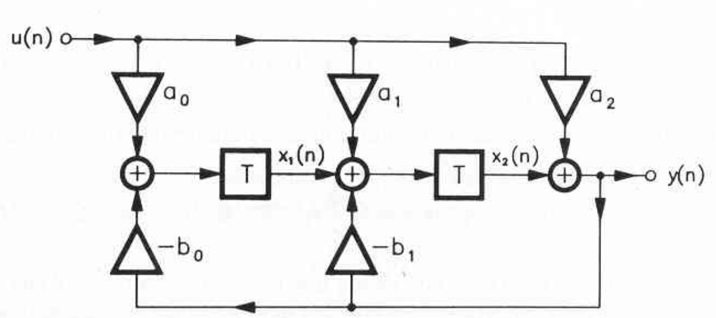


Bild 2.7: System 2. Ordnung

$$\begin{pmatrix} x_1(n+1) \\ x_2(n+1) \end{pmatrix} = \begin{pmatrix} 0 & -b_0 \\ 1 & -b_1 \end{pmatrix} \begin{pmatrix} x_1(n) \\ x_2(n) \end{pmatrix} + \begin{pmatrix} a_0 - a_2 b_0 \\ a_1 - a_2 b_1 \end{pmatrix} u(n) \tag{2.86}$$

$$y(n) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}^T \begin{pmatrix} x_1(n) \\ x_2(n) \end{pmatrix} + a_2 u(n). \tag{2.87}$$

$$\boldsymbol{x}(n+1) = \boldsymbol{A} \cdot \boldsymbol{x}(n) + \boldsymbol{b}\, u(n)$$
$$y(n) = \boldsymbol{c}^{\boldsymbol{T}} \cdot \boldsymbol{x}(n) + d\, u(n)$$

$$\boldsymbol{x}(n+1) = \boldsymbol{A} \cdot \boldsymbol{x}(n) + \boldsymbol{b}\, u(n)$$
$$y(n) = \boldsymbol{c}^{\boldsymbol{T}} \cdot \boldsymbol{x}(n) + d\, u(n)$$

Solution via unilateral Z-Transform:

$$z\,\boldsymbol{X}(z) - z\,\boldsymbol{x}[0] = \boldsymbol{A} \cdot \boldsymbol{X}(z) + \boldsymbol{b}\, U(z)$$
$$Y(z) = \boldsymbol{c}^{\boldsymbol{T}} \cdot \boldsymbol{X}(z) + d\, U(z)$$
$$Y_\rho(z) = \boldsymbol{c}^{\boldsymbol{T}} \cdot (z\boldsymbol{E} - \boldsymbol{A})^{-1} \boldsymbol{x}[0]\, z + [\boldsymbol{c}^{\boldsymbol{T}} \cdot (z\boldsymbol{E} - \boldsymbol{A})^{-1} \boldsymbol{b} + d]\, U_\rho(z)$$

Transfer Function:

$$H(z) = \frac{\boldsymbol{c}^{\boldsymbol{T}} \cdot (z\boldsymbol{E} - \boldsymbol{A})' \boldsymbol{b}}{det(z\boldsymbol{E} - \boldsymbol{A})} + d$$

$$Z^{-1}\{ det(z\boldsymbol{E} - \boldsymbol{A})\, Y(z) \} = Z^{-1}\{ [\boldsymbol{c}^{\boldsymbol{T}} \cdot (z\boldsymbol{E} - \boldsymbol{A})' \boldsymbol{b} + det(z\boldsymbol{E} - \boldsymbol{A})\, d]\, U(z) \}$$

Schedule Lecture 2: Deterministic Systems

- Systems Analysis and Design

Difference Equations

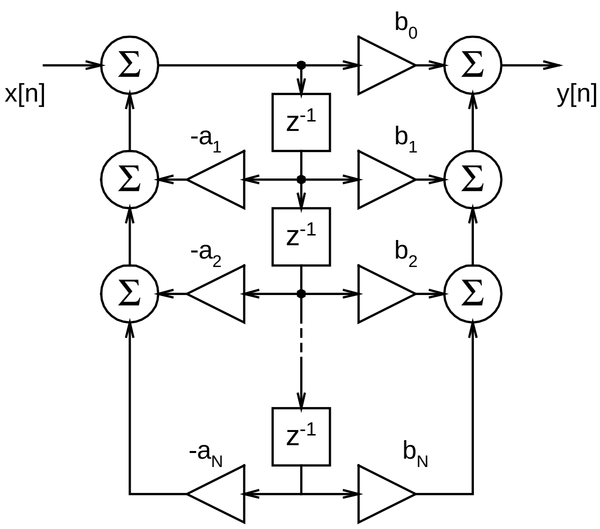
- The difference equation does not capture the initial sytem‘s condition.
- Initial conditions are introduced only by
  a) Unilateral Z-Transformation or
  b) State variables x[n]

Transfer function

- H(z) = Z{h[n]}
- H(z) = Y(z) / X(z)
- H(z) = y[n=0], if x[n] = z^n

Getting H(z) from the difference eq.

$$y_n + a_1\, y_{n\ \ 1} + a_2\, y_{n\ \ 2} + \ldots + a_N\, y_{n\ \ N} = b_0 x_m + b_1 x_{m\ \ 1} + b_2 x_{m\ \ 2} + \ldots + b_M\, x_m$$



$$y_n + a_1 y_{n-1} + a_2 y_{n-2} + \ldots + a_N y_{n-N} = b_0 x_m + b_1 x_{m-1} + b_2 x_{m-2} + \ldots + b_M x_{m-M}$$

$$Y(z)[1 + a_1 z^{-1} + a_2 z^{-2} + \ldots + a_N z^{-N}] = X(z)[b_0 + b_1 z^{-1} + b_2 z^{-2} + \ldots + b_M z^{-M}]$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \ldots + b_M z^{-M}}{1 + a_1 z^{-1} + a_2 z^{-2} + \ldots + a_N z^{-N}}$$

$$H(z) = z^{N-M} \frac{b_0 z^M + b_1 z^{M-1} + b_2 z^{M-2} + \ldots + b_M}{z^N + a_1 z^{N-1} + a_2 z^{N-2} + \ldots + a_N}$$

- **Now apply fundamental algebraic theorem**

$$H(z) = z^{N-M} \frac{b_0 z^M + b_1 z^{M-1} + b_2 z^{M-2} + \ldots + b_M}{z^N + a_1 z^{N-1} + a_2 z^{N-2} + \ldots + a_N}$$

- **E.g. numerator**

$$b_0 z^M + b_1 z^{M-1} + b_2 z^{M-2} + \ldots + b_M = b_0 \prod_{i=1}^{M} (z - z_{0i})$$

$$H(z) = b_0 z^{N-M} \frac{\prod_{i=1}^{M} (z - z_{0i})}{\prod_{i=1}^{N} (z - z_{\infty i})}$$

- **Pole-Zeroes form of the system transfer function**

Stability

- BIBO (bounded input – bounded output)

$$\sum_{k=-\infty}^{\infty} |h_k| < \infty$$

or:

- Poles of the System functions must lie in the unit circle |z|<1

Causality

- $$h[n]=0 \qquad \forall n<0$$

or:

- System Transfer Function H(z) must only have coefficients with n<=0

Allpass

- An allpass transfers all frequencies with the same amplitude magnification (i.e. only phases differ with frequency)
- Allpasses must have same order Q in numerator and denominator, and:

$$a_q = b^*_{Q-q} \quad \forall\, q=1..Q \qquad\qquad 1/z^*_{0\,q} = z_{\infty\, q} \quad \forall\, q=1..Q$$

- I.e. stable allpasses have zeroes out of the unit circle! (poles within)

Minimal phase systems

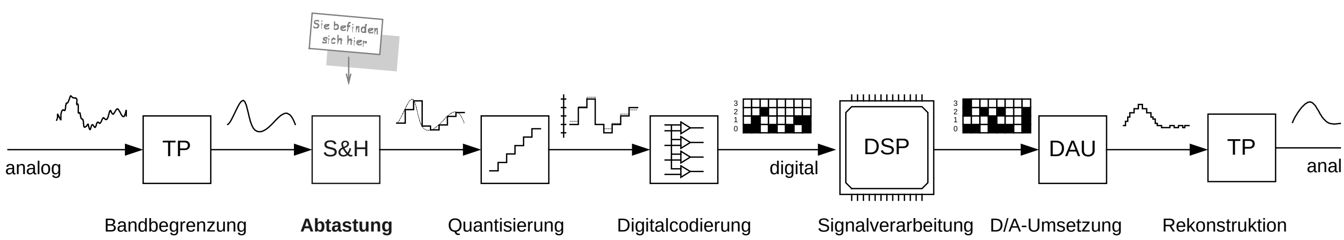
- (Stable) Systems with minimal phase shift over the frequency range:

They have all poles AND zeroes within the unit circle.

Schedule Lecture 2: Deterministic Systems

- System's response in the frequency domain

Sampling



Frequency transfer function

- Is periodic in multiples of $2\pi = \Phi = \omega T = \omega / f_A$

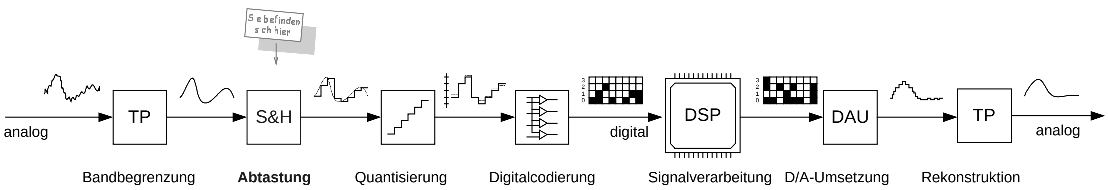
$$H(z) = H(e^{j\Phi k}) = \sum_{k=-\infty}^{\infty} h[k]\, e^{-j\Phi k} = H(e^{jk(\Phi + 2\pi n)})$$

$$h(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(e^{j\Phi})\, e^{j\Phi k}\, d\Phi$$

Sampling theorem

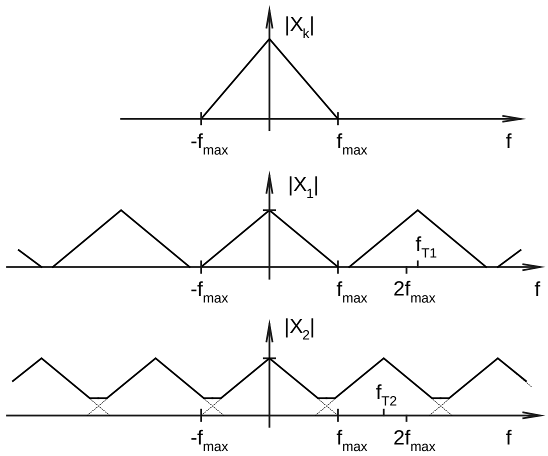
- Due to periodicity, spectra must not overlap.
- Hence: sampling frequency must be *twice* the highest frequency of the signal
- This actually also suffices to *reconstruct* the signal !! (sampling theorem)

- In order to guarantee this, have low-pass-filter before sampling the signal.

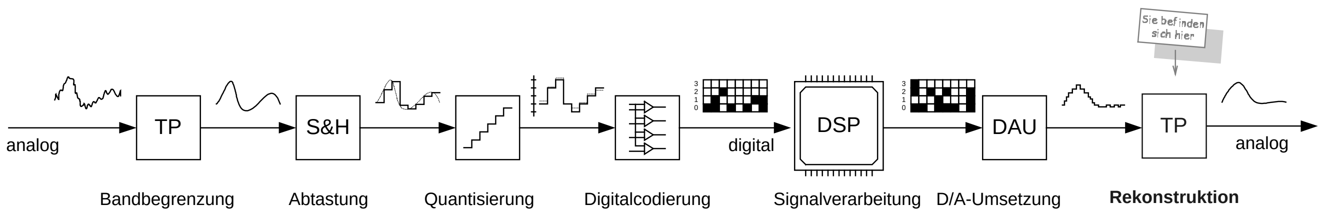


Aliasing

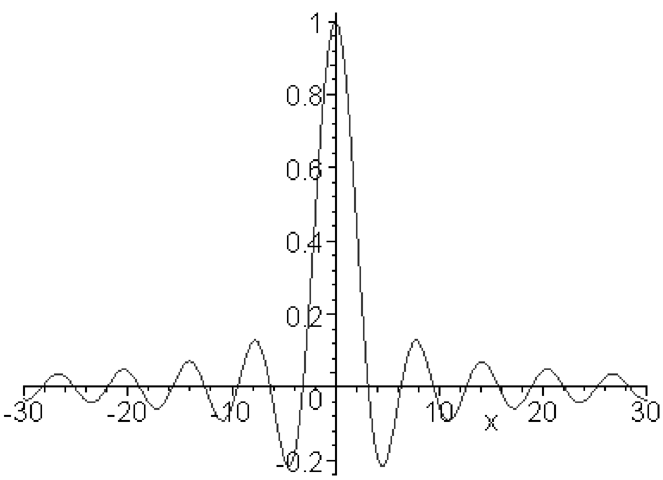
- **If the sampling theorem is not met, there is aliasing:**



Reconstruction



$$x(t) = \sum_{m=-\infty}^{\infty} x[m] \, sinc(t - mT)\pi / T$$



Reconstruction at finite window length

- **Window p(k)**
- **Special: rectangular window**

$$p(k)=\begin{cases}1 & 0\leq k<N\\ 0 & sonst\end{cases}$$

- **sampling:** $X_n^p(\omega)=\sum_{n-\infty}^{\infty} x[n]\,p[n]\,e^{\ j\omega nT}$
- **Reconstruction:**

$$x(t)=\sum_{m-\infty}^{\infty} x[m]\,p[m]\frac{sinc(t-mT)\pi}{T} \qquad X_n^p(\omega)=X_n(j\omega)*P(j\omega)$$

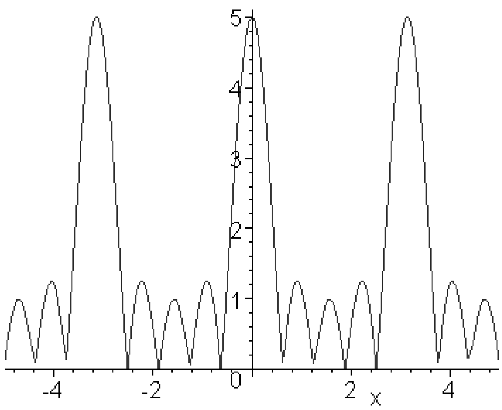
Reconstruction with rectangular window

- **Is convolution with window spectrum:**

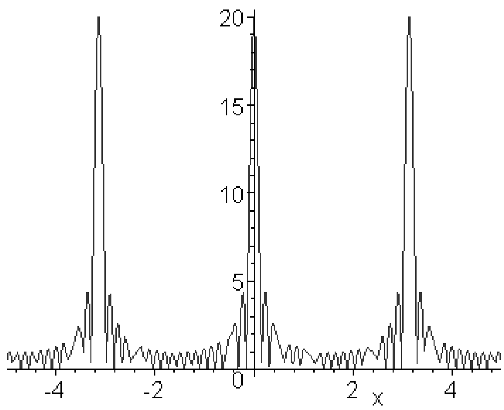
$$X_n^p(\omega)=X_n(j\omega)*P(j\omega)$$

P(j w):

N=5



N=20



Schedule Lecture 2: Deterministic Systems

- Discrete /fast Fourier transforms

DFT: discrete frequencies

- We had: Discrete time:

$$X(j\omega) = \sum_{k=-\infty}^{\infty} x[k] \cdot \mathrm{e}^{-j\omega kT}$$

$$x[k] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(j\omega) \cdot \mathrm{e}^{j\omega kT}\, d(\omega T)$$

- Now: discrete time & discrete frequencies:

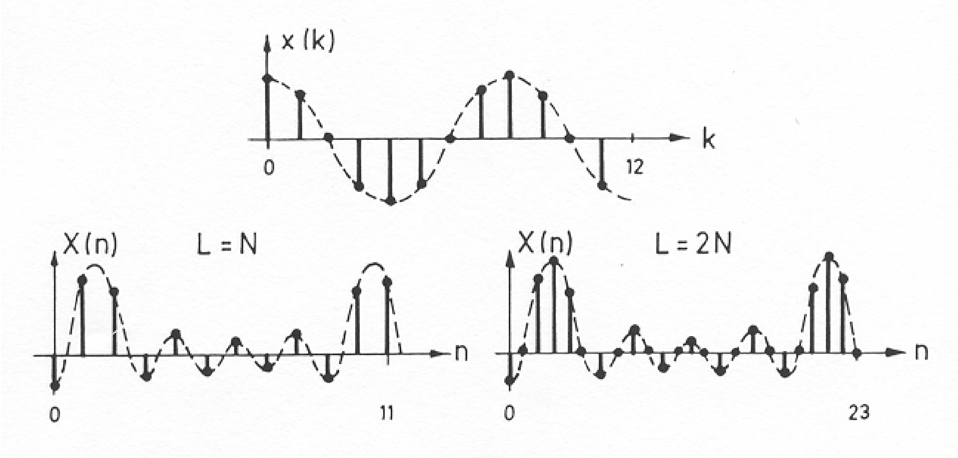
$$\omega = n\,\omega_0 = n\frac{\Omega}{N} = n\frac{2\pi}{NT} \quad ; \text{ beachte } \Omega = \frac{2\pi}{T}$$

DFT: $$X(n) = \sum_{k=0}^{N-1} x[k] \cdot W_N^{nk} = \sum_{k=0}^{N-1} x[k] \cdot \mathrm{e}^{-j\frac{2\pi}{N}\cdot k \cdot n}$$

IDFT: $$x[k] = \sum^{N-1} X(n) \cdot W_N^{-nk} = \sum^{N-1} X(n) \cdot \mathrm{e}^{j\frac{2\pi}{N}\cdot k \cdot n}$$

Oversampling

- Say, the original signal is uniquely sampled by L values (sampling theorem, !)
- Then N=L values of the discrete time and discrete frequency reconstruct the signal!
- If more than N values are known (oversampling), no information is gained



Windows

- Effect of windows: convolution of spectrum

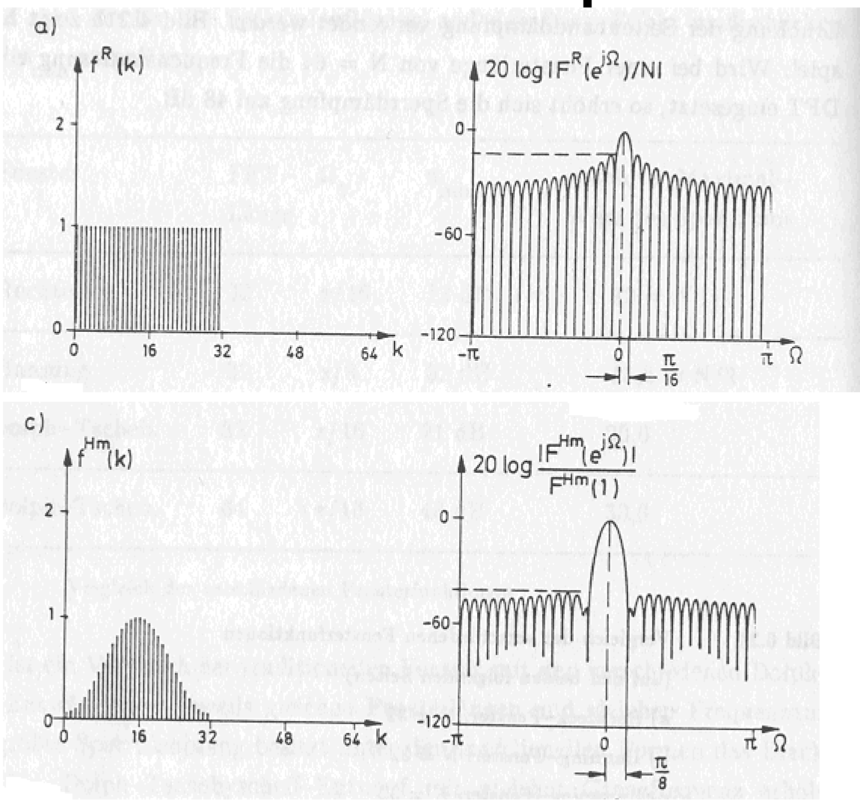
$$X_n^p(\omega) = X_n(j\omega) * P(j\omega)$$

- I.e. need to find window with „nice" spectrum P
- Look at rectangular window versus Hamming window

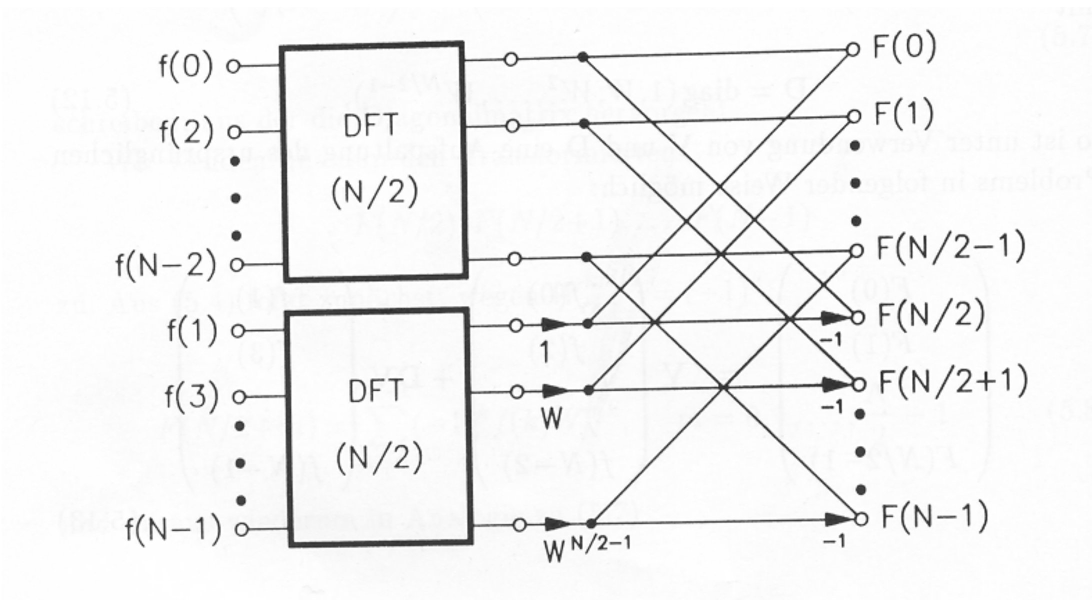
$$w[k] = \begin{cases} 0.54 + 0.56\cos(2\pi \frac{k}{N-1}) & \text{für } 0 \leqslant k \leqslant N-1 \\ 0 & \text{sonst} \end{cases}$$

- 32-point window dubbing:
  13 dB (rect), 42 dB (Hamming)

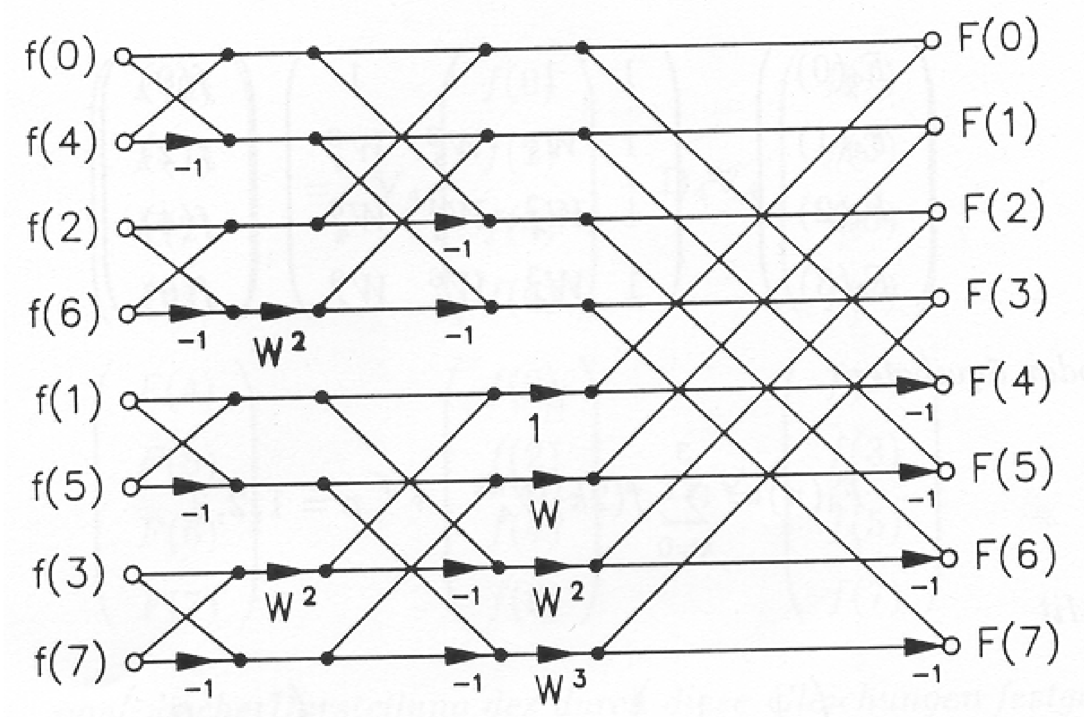
Window spectra



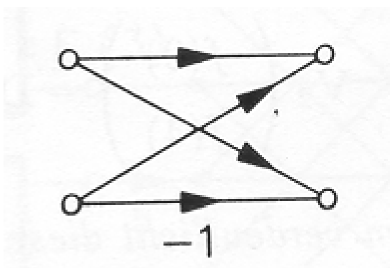
Fast Fourier Transfornation



FFT cascade



FFT basic module („butterfly“)

- 

extremely simple

If we have N sampling values, and if N is a power of 2, we need only log2(N) many butterfly operations: FAST !

- This holds for the Inverse FFT as well.
- If N not a power of 2, append zeroes.

Review: Deterministic Systems

- Refresh
- Systems Analysis and Design
- System's response in the frequency domain
- Discrete /fast Fourier transforms